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Enriched Quantales Arising from Complete Orthomodular Lattices

Abstract. This paper connects complete orthomodular lattices to two enriched quantale structures. Complete orthomodular lattices emphasize a static perspective of a quantum system, helping us reason about testable properties of a quantum system. Quantales offer a dynamic perspective, helping us reason about the structure of quantum actions. We enrich quantales with an orthocomplementation-inducing operator, and call these structures orthomodular dynamic algebras. One type of orthomodular dynamic algebra distinguishes the joins of any two different sets of atoms, while the other distinguishes elements by the collective behavior of the atoms below it. We show that both orthomodular dynamic algebras are unital, and the unit is the top element of an induced orthomodular lattice. We provide a categorical equivalence between both orthomodular dynamic algebras and complete orthomodular lattices with isomorphisms, and we show that this equivalence is preserved when augmenting the orthomodular dynamic algebras with an involution. These equivalences help clarify the relationship between static and dynamic quantum structures.

Keywords: Orthomodular dynamic algebra, Complete orthomodular lattice, Quantale.

1. Introduction

Much of our reasoning about quantum systems involves quantum dynamics. In quantum computation, unitary operators may prepare a quantum state for measurement, and projectors represent the effect of certain quantum measurements. A very general tool for capturing quantum dynamics is the quantale. While it emphasizes certain algebraic properties of a quantum system, such as the potential non-commutativity of quantum actions, they do not by themselves isolate roles of particular quantum actions such as unitaries and projectors. By enriching a quantale with an orthocomplementation-inducing operator, we obtain a collection of elements whose behavior is that of projectors. In [2,5], this was done by taking the set

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of projectors to be the image of an orthocomplementation-inducing operator. When the projectors form an orthomodular lattice, such enriched quantales are called orthomodular dynamic algebras. The paper [5] also established a categorical equivalence between a category of orthomodular dynamic algebras and the category of complete orthomodular lattices with bijective ortholattice morphisms; to construct an orthomodular dynamic algebra from a complete orthomodular lattice, the elements of the orthomodular dynamic algebra were taken to be sets of compositions of Sasaki projections on the complete orthomodular lattice.

While the orthomodular dynamic algebra in [5] was able to isolate projectors, it could not account for unitary operators. Unitary operators are important in quantum computation. They allow us to transform a state that starts off in the standard basis to one that is in superposition. Preparing a state via unitary evolutions, which is effectively changing the basis or frame of the state, enables us to more easily measure using a convenient basis. In this paper, we develop two new orthomodular dynamic algebras. One, which we call F-ODA, expands the orthomodular dynamic algebra of [5] with unitary operators; it identifies each element with the set of atoms below it. The other, which we call R-ODA, also involves unitary operators, but only distinguishes elements that have distinct collective behavior of the atoms below it. We show that both are atomistic and unital.

While quantum dynamic algebras provide a dynamic perspective of a quantum system, complete orthomodular lattices provide a static perspective, viewing points as testable properties. Orthomodular lattices are central to early static treatments of quantum logic and play an important role in more recent dynamic quantum logics. For both of our orthomodular dynamic algebras, we establish a representation theorem linking orthomodular dynamic algebras with complete orthomodular lattices. An F-ODA is constructed from a complete orthomodular lattice by taking sets of compositions of Sasaki projectors and ortholattice automorphisms. An R-ODA is constructed from a complete orthomodular lattice by taking its elements to be relations, where each such relation is the union of the members of an element of an F-ODA; the quantale component of a constructed R-ODA is a "relational quantale" from [3], but where the join is set union. Moreover, we expand the representation theorem into a categorical equivalence between both orthomodular dynamic algebras and complete orthomodular lattices with isomorphisms. Such representation theorems and categorical equivalences help us relate the static and dynamic algebraic structures.

The paper is organized as follows. In Section 2, we define complete orthomodular lattices and introduce F-ODAs and R-ODAs; we show that both orthomodular dynamic algebras are atomistic and unital. In Section 3, we provide constructions of an F-ODA and R-ODA from a complete orhomodular lattice. In Section 4, we prove a representation theorem connecting each orthomodular dynamic algebra to complete orthomodular lattices. Then in Section 5, we prove a categorical equivalence between both orthomodular dynamic algebras and complete orthomodular lattices with isomorphisms. In Section 6, we expand orthomodular dynamic algebras with an involution and show how we can maintain equivalence of the corresponding structures. We conclude with Sections 7 and 8.

2. Complete Orthomodular Lattices and Orthomodular Dynamic Algebras

2.1. Complete Orthomodular Lattices

DEFINITION 2.1. Consider a tuple $\mathfrak{L} = (L, \leq, -^{\perp})$ where L is a non-empty set, $\leq \subseteq L \times L$ is a partial order and $-^{\perp} : L \to L$ is a function. It is an *ortho-lattice*, if it satisfies (1) - (3) below. It is a *complete orthomodular lattice*, abbreviated COL, if it satisfies (1) - (5).

1. $(L, \leq)$ is a lattice: any $p, q \in L$ have a least upper bound $p \vee q$ in L, called the join, and a greatest lower bound $p \wedge q$ in L, called the meet;
2. there are $\mathbb{0}, \mathbb{1} \in L$ such that $\mathbb{0} \leq p \leq \mathbb{1}$, for each $p \in L$;
3. $-^{\perp}$ is an orthocomplementation, i.e. for any $p, q \in L$;
   (a) $p \wedge p^{\perp} = \mathbb{0}$ and $p \vee p^{\perp} = \mathbb{1}$;
   (b) $p \leq q \Rightarrow q^{\perp} \leq p^{\perp}$;
   (c) $p^{\perp\perp} = p$;
4. $(L, \leq)$ is a complete lattice: for any $\{p_i \in L \mid i \in I\}$, they have a join $\bigvee\{p_i \in L \mid i \in I\}$ and a meet $\bigwedge\{p_i \in L \mid i \in I\}$ in L;
5. orthomodularity holds, i.e. for any $p, q \in L$, $p \leq q$ implies that $p = q \wedge (p \vee q^{\perp})$.

In an ortho-lattice $\mathfrak{L} = (L, \leq, -^{\perp})$, for each $p \in L$, we can define an important pair of operations called the *Sasaki projection (onto* p) and *Sasaki hook (from* p) [6]:

$$f_p : L \to L :: q \mapsto p \wedge (p^{\perp} \vee q), \qquad f^p : L \to L :: q \mapsto p^{\perp} \vee (p \wedge q).$$

A crucial fact about this pair of order-preserving maps is that $\mathfrak{L}$ is orthomodular (5) if and only if, for every $p \in L$, f_p is left adjoint to f^p [4]. Therefore, in an orthomodular lattice, Sasaki projections preserve (arbitrary) joins.

LEMMA 2.2. *In a complete orthomodular lattice* $\mathfrak{L} = (L, \leq, -^{\perp})$*, for any* $p \in L$ *and* $K \subseteq L$*,* $f_p(\bigvee_{q \in K} q) = \bigvee_{q \in K} f_p(q)$*.*

DEFINITION 2.3. An *ortho-lattice isomorphism*, or $\mathbb{L}$*-morphism*, from an ortho-lattice $\mathfrak{L}_1 = (L_1, \leq_1, -^{\perp_1})$ to $\mathfrak{L}_2 = (L_2, \leq_2, -^{\perp_2})$ is a function $k : L_1 \to L_2$ such that, for any $p_1, q_1 \in L_1$,

6. k is a bijection;
7. $p_1 \leq_1 q_1 \Leftrightarrow k(p_1) \leq_2 k(q_1)$;
8. $k(p_1^{\perp_1}) = (k(p_1))^{\perp_2}$.

LEMMA 2.4. *For any ortho-lattice isomorphism* k *from an ortho-lattice* $\mathfrak{L}_1 = (L_1, \leq_1, -^{\perp_1})$ *to* $\mathfrak{L}_2 = (L_2, \leq_2, -^{\perp_2})$ *and any* $p \in L_1$*,* $k \circ f_p = f_{k(p)} \circ k$*.*

PROOF. For any $q \in L_1$, $(k \circ f_p)(q) = k(p \wedge (p^{\perp} \vee q)) = (k(p) \wedge (k(p)^{\perp} \vee k(q))) = f_{k(p)}(k(q)) = (f_{k(p)} \circ k)(q)$. ■

2.2. Two Orthomodular Dynamic Algebras

In this subsection, we define two orthomodular dynamic algebras, one that we refer to as an F-ODA, whose elements are abstractions of sets of operators, and the other as an R-ODA, whose elements are abstractions of relations. Roughly speaking, an orthomodular dynamic algebra is an enrichment of a quantale. We first recall the definition of a quantale and introduce some other notions, and then we give the definition of the orthomodular dynamic algebra.

DEFINITION 2.5. A *quantale* is a tuple $(Q, \sqsubseteq, \cdot)$, such that

9. $(Q, \sqsubseteq)$ is a complete lattice, that is, a partially ordered set, where every (potentially infinite) subset A has a least upper bound $\bigsqcup A$ called the *join* of A[1];
10. $\cdot$ is a binary operation on Q that is associative: for any $x, y, z \in Q$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

[1] The existence of a join for every (potentially infinite) set guarantees the existence of a meet as well; the meet (greatest lower bound) of A is the join of the set of lower bounds of A.

11. $\cdot$ distributes over $\bigsqcup$: for any $x \in Q$ and $K \subseteq Q$,

$$x \cdot \bigsqcup K = \bigsqcup\{x \cdot y \mid y \in K\} \text{ and } \left(\bigsqcup K\right) \cdot x = \bigsqcup\{y \cdot x \mid y \in K\};$$

In what follows, we denote by $\wp$ the power set operation.

EXAMPLE 2.6. (*Quantale as a set of sets of functions*) Let A be a non-empty set and let F be a set of functions from A to A that is closed under composition. Then let $\mathfrak{Q} = (Q, \sqsubseteq, \cdot)$, where $Q = \wp(F)$, $\sqsubseteq$ is set inclusion $\subseteq$, and $\cdot$ is pairwise composition, such that $A \cdot B = \{a \circ b \mid a \in A, b \in B\}$. Then $\mathfrak{Q}$ is a quantale.

EXAMPLE 2.7. (*Quantale as a set of relations*) Let A be a set and let Q be a set of binary relations on A closed under relation composition and union. Then $\mathfrak{Q} = (Q, \subseteq, \cdot)$ is a quantale (where $\cdot$ is relation composition)

DEFINITION 2.8. A *generalized dynamic algebra* is a tuple $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ such that Q is a non-empty set, and all of $\bigsqcup : \wp(Q) \to Q$, $\cdot : Q \times Q \to Q$ and $\sim : Q \to Q$ are functions.

An element $x \in Q$ is *invertible* if there exists $y \in Q$ such that for all $z \in Q$, $(x \cdot y) \cdot z = z \cdot (x \cdot y) = (y \cdot x) \cdot z = z \cdot (y \cdot x) = z$. The following are some constructions on a generalized dynamic algebra.

$$\begin{aligned}
\sqsubseteq &\stackrel{\text{def}}{=} \left\{(x, y) \in Q \times Q \mid \bigsqcup\{x, y\} = y\right\} \\
\mathcal{P}_{\mathfrak{Q}} &\stackrel{\text{def}}{=} \{\sim x \mid x \in Q\} \\
\bigvee X &\stackrel{\text{def}}{=} \sim\sim \bigsqcup X, \text{ for any } X \subseteq \mathcal{P}_{\mathfrak{Q}} \\
\bigwedge X &\stackrel{\text{def}}{=} \sim \bigsqcup \{\sim x \mid x \in X\}, \text{ for any } X \subseteq \mathcal{P}_{\mathfrak{Q}} \\
\preceq &\stackrel{\text{def}}{=} \left\{(p, q) \in \mathcal{P}_{\mathfrak{Q}} \times \mathcal{P}_{\mathfrak{Q}} \mid \bigvee\{p, q\} = q\right\} \\
\ulcorner x \urcorner &\stackrel{\text{def}}{=} \lambda y. \sim\sim(x \cdot y) \\
\equiv &\stackrel{\text{def}}{=} \{(x, y) \in Q \times Q \mid \ulcorner x \urcorner(p) = \ulcorner y \urcorner(p), \text{ for each } p \in \mathcal{P}_{\mathfrak{Q}}\} \\
\mathcal{U}_{\mathfrak{Q}} &\stackrel{\text{def}}{=} \{x \in Q \mid x \text{ is invertible} \\
&\qquad \text{and} \ulcorner x \urcorner \text{restricted to} \mathcal{P}_{\mathfrak{Q}} \text{is a permutation of} \mathcal{P}_{\mathfrak{Q}} \text{that preserves} \bigvee \text{and} \sim\} \\
\mathcal{T}_{\mathfrak{Q}} &\stackrel{\text{def}}{=} \{x \in Q \mid x = p_1 \cdot \cdots \cdot p_n, \text{ for some } n \in \mathbb{N}^+ \text{ and } p_1, \ldots, p_n \in \mathcal{P}_{\mathfrak{Q}} \cup \mathcal{U}_{\mathfrak{Q}}\}
\end{aligned}$$

Note that $\mathcal{T}_{\mathfrak{Q}}$ is the smallest subset of Q containing $\mathcal{P}_{\mathfrak{Q}} \cup \mathcal{U}_{\mathfrak{Q}}$ which is closed under the operation $\cdot$.

In the following, for simplicity, we write $x \sqcup y$ for $\bigsqcup\{x, y\}$, $x \vee y$ for $\bigvee\{x, y\}$ and $x \wedge y$ for $\bigwedge\{x, y\}$; and we may omit the subscripts in $\mathcal{P}_{\mathfrak{Q}}$ and $\mathcal{T}_{\mathfrak{Q}}$.

Definition 2.9. A *Relation-based orthomodular dynamic algebra*, abbreviated R-ODA, is a generalized dynamic algebra $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ satisfying all the conditions below except 22; a *Function-based orthomodular dynamic algebra*, abbreviated F-ODA, is a generalized dynamic algebra satisfying all the conditions below except 21:

12. $(Q, \sqsubseteq, \cdot)$ is a quantale, and $\bigsqcup$ is the arbitrary join.[2]
13. $(\mathcal{P}, \preceq, \sim)$ is a complete orthomodular lattice;
14. For every ortholattice isomorphism $f : \mathcal{P} \to \mathcal{P}$, there exists an $a \in \mathcal{U}$, such that $f = \ulcorner a \urcorner$ (maximality);
15. If X is such that
    (a) $\mathcal{P} \cup \mathcal{U} \subseteq X \subseteq Q$,
    (b) X is closed under the operation $\cdot$, and
    (c) X is *closed under* $\bigsqcup$, by which we mean, for any $\mathcal{X} \in \wp(X)$, $\bigsqcup \mathcal{X} \in X$.

    Then $X = Q$ (minimality);
16. for any $x, y \in \mathcal{T}$, $x = y$ if and only if $x \equiv y$ (completeness);
17. for any $p, q \in \mathcal{P}$, $\ulcorner p \urcorner(q) = f_p(q)$, i.e. $\sim\sim(p \cdot q) = p \wedge (\sim p \vee q)$ (Sasaki projection);
18. for any $a \in \mathcal{U}$ and $p \in \mathcal{P}$, $a \cdot p = \ulcorner a \urcorner(p) \cdot a$ (conjugate).
19. $\ulcorner x \urcorner(y) = \ulcorner x \urcorner(\sim\sim y)$, for each $x, y \in Q$ (composition).
20. $\sim \bigsqcup \{x_i \mid i \in I\} = \sim \bigsqcup \{\sim\sim x_i \mid i \in I\}$, whenever $\{x_i \mid i \in I\} \subseteq \mathcal{T}$ (join)
21. For any $X, Y \subseteq \mathcal{T}$, $\bigsqcup X = \bigsqcup Y$ if and only if $\bigcup \{\ulcorner x \urcorner \mid x \in X\} = \bigcup \{\ulcorner x \urcorner \mid x \in Y\}$ (relation);
22. for any $X, Y \subseteq \mathcal{T}$, $\bigsqcup X = \bigsqcup Y$, if and only if $X = Y$ (sets);

We write $\mathfrak{Q}^R$ and $\mathfrak{Q}^F$ for relation and function based orthomodular dynamic algebras respectively.

The F-ODA is similar to the orthomodular dynamic algebra defined in [5], except for the involvement of unitaries $\mathcal{U}$ as well as the addition of properties (14) (maximality), (18) (conjugate) and (20) (join). The property (maximality) is useful for ensuring that the F-ODA or R-ODA has as many unitaries as is naturally constructed by our map from COL to ODA. The property (conjugate) is a unitary analogue of (17), and relates $\ulcorner a \urcorner$ to a.

[2] The original function $\bigsqcup$ is then the join operation on the lattice structure, from which the order relation can be recovered by $x \sqsubseteq y$ iff $x \sqcup y = y$.

The property (join) is included to help establish an orthocomplementation-preserving map in the proof of the representation theorem (Theorem 4.2 ahead); although it does not involve unitaries, it was not included in [5] since in the absence of unitaries, ODA morphisms did not need to preserve orthocomplementation in order to establish caterogical equivalence.

REMARK 2.10. (*Difference between F-ODAs and R-ODAs*) This remark shows that F-ODAs and R-ODAs are incomparable (there are F-ODAs that are not R-ODAs and there are R-ODAs that are not F-ODAs).

Let $\mathfrak{L}$ be the lattice of subspaces of $\mathbb{R}^2$. Let $\mathcal{P}$ be the set of all Sasaki projectors on $\mathfrak{L}$ and $\mathcal{U}$ be the set of all ortholattice automorphisms on $\mathfrak{L}$. Let $\mathcal{F}$ be the smallest set containing $\mathcal{P}$ and $\mathcal{U}$ and is closed under function composition. Let

$$\mathcal{Q}^R = \{\bigcup A \mid A \in \wp(\mathcal{F})\}$$
$$\mathcal{Q}^F = \wp(\mathcal{F})$$

Then $\mathcal{Q}^R$ equipped with arbitrary union and relation composition forms a quantale as a set of relations, which we will see can be extended to an R-ODA $\mathbf{F}^R(\mathfrak{L})$ according to Theorem 3.18. Similarly, $\mathcal{Q}^F$ equipped with arbitrary union and pairwise composition forms a quantale as a set of sets of functions, which we will see can be extended to an F-ODA $\mathbf{F}^F(\mathfrak{L})$ according to Theorem 3.18. In $\mathbf{F}^R(\mathfrak{L})$, $\mathcal{T}^R$ is just $\mathcal{F}$; in $\mathbf{F}^F(\mathfrak{L})$, $\mathcal{T}^F$ is $\{\{a\} \mid a \in \mathcal{F}\ \}$.

Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (rotation by $\pi/4$, acting on subspaces). We focus on how U transforms subspaces of $\mathbb{R}^2$ (that is, it is an operator on $\mathfrak{L}$). Since Uv is never orthogonal to v when $v \neq 0$, we have equality of relations:

$$\bigcup \mathcal{P} = U \cup \bigcup \mathcal{P}$$

In other words, every result of mapping by U can be matched by a projector in $\mathcal{P}$.

This illustrates that

while $\mathbf{F}^R(\mathfrak{L})$ is an R-ODA (by the anticipated Section 3 construction and Theorem 3.18) and thus satisfies (21), $\mathbf{F}^R(\mathfrak{L})$ allows distinct sets of elements of $\mathcal{T}^R$ have the same join; hence $\mathbf{F}^R(\mathfrak{L})$ does not satisfy (22). Thus the R-ODA $\mathbf{F}^R(\mathfrak{L})$ cannot be an F-ODA.

By the proof of Proposition 3.16, we will see that

$$\bigcup\{\ulcorner x \urcorner^F \mid x \in \mathcal{P}\} = \bigcup\{\ulcorner y \urcorner^F \mid y \in \{U\} \cup \mathcal{P}\}.$$

while $\mathcal{P} \neq \{U\} \cup \mathcal{P}$. Thus while $\mathbf{F}^F(\mathfrak{L})$ is an F-ODA and thus satisfies (22), $\mathbf{F}^F(\mathfrak{L})$ requires distinct sets of elements of $\mathcal{T}^F$ have distinct joins even when those sets of elements of $\mathcal{T}^F$ exhibit the same behavior, and hence does not satisfy (21). Hence the F-ODA $\mathbf{F}^F(\mathfrak{L})$ cannot be an R-ODA.

Remark 2.11. (*Sasaki projection, Sasaki hook, and Orthomodularity*) By (17) Sasaki projection can be expressed using the primitives of an ODA: $f_p(q) = \sim\sim(p \cdot q)$, for any $p, q \in \mathcal{P}$.

We obtain a similar expression for Sasaki hook as follows: Applying $\sim$ to both sides of the above equation we get $\sim f_p(q) = \sim\sim\sim(p \cdot q)$. For the left-hand side, $\sim f_p(q) = \sim(p \wedge (\sim p \vee q)) = \sim p \vee (p \wedge \sim q) = f^p(\sim q)$; and the right-hand side can be simplified to $\sim(p \cdot q)$. Therefore, we have $f^p(q) = \sim(p \cdot \sim q)$, for any $p, q \in \mathcal{P}$, which is the same as that on Page 2277 in [2].

Orthomodularity can be characterized by the inequality $p \wedge f^p(q) \leq q$, which is equivalent to the equation $p \wedge f^p(q) = p \wedge q$. Combining this with the above observation, orthomodularity can be characterized as: for any $p, q \in \mathcal{P}$, $p \wedge \sim(p \cdot \sim q) = p \wedge q$, i.e.

$$\sim\bigsqcup\{\sim p, \sim\sim(p \cdot \sim q)\} = \sim\bigsqcup\{\sim p, \sim q\}.$$

This shows an interesting interplay between the primitives $\bigsqcup$, $\cdot$ and $\sim$. Item (17) itself is also an interesting interplay between the primitives, which is

$$\sim\bigsqcup\left\{\sim p, \sim\bigsqcup\{\sim p, q\}\right\} = \sim\sim(p \cdot q),$$

for any $p, q \in \mathcal{P}$. Finally, to do without $\mathcal{P}$, orthomodularity becomes: for any $x, y \in \mathcal{Q}$,

$$\sim\bigsqcup\{\sim\sim x, \sim\sim(\sim x \cdot \sim\sim y)\} = \sim\bigsqcup\{\sim\sim x, \sim\sim y\}.$$

And Item (17) becomes: for any $x, y \in \mathcal{Q}$,

$$\sim\bigsqcup\left\{\sim\sim x, \sim\bigsqcup\{\sim\sim x, \sim y\}\right\} = \sim\sim(\sim x \cdot \sim y).$$

2.2.1. Observable behavior We view the elements of the $\mathcal{P}$ as projectors, and as is common in quantum logic, each projector may be identified by an observable property. For a concrete example, the image of a projector on a Hilbert space is a closed linear subspace of the Hilbert space, and such subspaces are often called "testable properties". For each ODA element $x \in Q$, we view the function $\ulcorner x \urcorner$ as the "observable behavior" of x, that is the behavior of x acting on elements $y \in Q$ via $\sim\sim(x \cdot y)$. We usually take $y \in \mathcal{P}$, making $\ulcorner x \urcorner$ an operator on the set $\mathcal{P}$ of projectors of the ODA.

The following lemma states that the behavior of a semigroup product is the function composition of the behaviors. The result is similar to [5, Lemma

3], but accounts for unitaries and applies to the behavior of any quantale element rather than just those in $\mathcal{P}$; although the proof is similar, we include one here for completeness.

LEMMA 2.12. *Let* $(Q, \bigsqcup, \cdot, \sim)$ *be an* F*-ODA or an* R*-ODA. For any* $n \in \mathbb{N}^+$ *and* $x_1, \ldots, x_n \in Q$, $\ulcorner x_1 \cdot \cdots \cdot x_n \urcorner = \ulcorner x_1 \urcorner \circ \cdots \circ \ulcorner x_n \urcorner$.

As a special case where $p_i \in \mathcal{P}$, this lemma states that $\ulcorner p_1 \cdot \cdots \cdot p_n \urcorner = \ulcorner p_1 \urcorner \circ \cdots \circ \ulcorner p_n \urcorner = f_{p_1} \circ \cdots \circ f_{p_n}$.

PROOF. We use induction on n to show that $\ulcorner x_1 \cdot \cdots \cdot x_n \urcorner = \ulcorner x_1 \urcorner \circ \cdots \circ \ulcorner x_n \urcorner$. The base case ($n = 1$) is trivial. For the induction step, our induction hypothesis (IH) is that the statement holds for any collection of elements of length $n = k$. Let $n = k + 1$. For each $y \in Q$,

$$\begin{aligned}
\ulcorner x_1 \cdot \cdots \cdot x_k \cdot x_{k+1} \urcorner(y) &= \sim\sim(x_1 \cdot \cdots \cdot x_k \cdot x_{k+1} \cdot y) \\
&= \ulcorner x_1 \urcorner(x_2 \cdot \cdots \cdot x_k \cdot x_{k+1} \cdot y) \\
&= \ulcorner x_1 \urcorner(\sim\sim(x_2 \cdot \cdots \cdot x_k \cdot x_{k+1} \cdot y)) \quad (\text{by}(19)) \\
&= \ulcorner x_1 \urcorner(\ulcorner x_2 \cdot \cdots \cdot x_k \cdot x_{k+1} \urcorner(y)) \\
&= \ulcorner x_1 \urcorner((\ulcorner x_2 \urcorner \circ \cdots \circ \ulcorner x_k \urcorner \circ \ulcorner x_{k+1} \urcorner)(y))(\text{IH}) \\
&= (\ulcorner x_1 \urcorner \circ \cdots \circ \ulcorner x_k \urcorner \circ \ulcorner x_{k+1} \urcorner)(y)
\end{aligned}$$

This finishes the proof by induction. ■

2.2.2. Atomicity and Normal Form We show that the quantale lattice structure of an orthomodular dynamic algebra (either an F-ODA or an R-ODA) is atomistic, that is each element x is the join of a set of atoms. We will see that $\mathcal{T}$ is the set of all quantale atoms. For an F-ODA, property (22) ensures that the set of atoms whose join is x is unique. For an R-ODA, we saw from Remark 2.10 that this set of atoms is not necessarily unique: there may be different sets of atoms with the same join x (thus F-ODAs and R-ODAs are distinct). However (21) guarantees the uniqueness of the collective "behaviors" of the atoms whose join is a given element. For an F-ODA or an R-ODA, any set of atoms whose join is x is always a subset of the set of atoms below x and the set of all atoms below x will have x as its join. Expressing an element as the join of all atoms below serves as a normal form.

The following lemma asserts that, assuming that $\mathcal{T}$ is the set of quantale atoms (Lemma 2.16 will show this indeed is the case), both F-ODAs and R-ODAs are atomistic. The proof is similar to the one for [5, Lemma 2], but accounts for unitaries as well; although similar, we include a proof here with some key steps for completeness.

LEMMA 2.13. *Let* $(Q, \bigsqcup, \cdot, \sim)$ *be an* F*-ODA or an* R*-ODA. For each* $x \in Q$*, there exists a set* $A \subseteq \mathcal{T}$ *such that* $x = \bigsqcup A$.

PROOF. Let $Z \subseteq Q$ consist of each x in Q for which there is a set $A \subseteq \mathcal{T}$ such that $x = \bigsqcup A$. We will show that $Z = Q$ by showing that Z satisfies each of the three conditions in (15).

- Clearly $\mathcal{P} \cup \mathcal{U} \subseteq \mathcal{T}$ and $\mathcal{T} \subseteq Z$, and hence $\mathcal{P} \cup \mathcal{U} \subseteq Z$.
- Suppose $y, z \in Z$ and $x = y \cdot z$. We wish to show that $x \in Z$. Toward this, observe that $y = \bigsqcup Y$ and $z = \bigsqcup V$ for some $Y, V \subseteq \mathcal{T}$. Then by (11),

$$x = y \cdot z = \bigsqcup Y \cdot \bigsqcup V = \bigsqcup \{y \cdot v \mid (y, v) \in Y \times V\}.$$

- Suppose $V \in \wp(Z)$ and $x = \bigsqcup V$. We want to show that $x \in Z$. Toward this, observe that for all $v \in V$, there exists $Y_v \subseteq \mathcal{T}$, such that $v = \bigsqcup Y_v$. Then

$$x = \bigsqcup V = \bigsqcup \{\bigsqcup Y_v \mid v \in V\} = \bigsqcup \{a \mid a \in Y_v, v \in V\}.$$

  Note that $\{a \mid a \in Y_v, v \in V\} \subseteq \mathcal{T}$, since each $Y_v \subseteq \mathcal{T}$.

Thus by (15), $Z = Q$, i.e., the desired existence property holds for every $x \in Q$. ■

REMARK 2.14. Lemma 2.13 shows that each $x \in Q$ can be decomposed into a join of elements of $\mathcal{T}$. For an F-ODA, uniqueness of the set of elements from $\mathcal{T}$ whose join is a given element is given by the left-to-right direction of (22): If $A, B \subseteq \mathcal{T}$ such that $\bigsqcup A = \bigsqcup B$, then $A = B$.

For an R-ODA, the set of such elements from $\mathcal{T}$ need not be unique, however according to the left-to-right direction of (21), the collective "behaviors" of elements from $\mathcal{T}$ whose join is a given element is uniquely determined: If $A, B \subseteq \mathcal{T}$ such that $\bigsqcup A = \bigsqcup B$, then $\bigcup\{\ulcorner a \urcorner \mid a \in A\} = \bigcup\{\ulcorner b \urcorner \mid b \in B\}$.

Toward showing that $\mathcal{T}$ is the set of quantale atoms, we first observe that no element of $\mathcal{T}$ is above any other element of $\mathcal{T}$ in the quantale ordering.

LEMMA 2.15. *Let* $\mathfrak{Q}$ *be an* F*-ODA or an* R*-ODA. For any* $a, b \in \mathcal{T}$*, if* $a \sqsubseteq b$*, then* $a = b$.

PROOF. Let $(Q, \bigsqcup, \cdot, \sim)$ be an R-ODA (the case where it is an F-ODA is simpler). Suppose $a, b \in \mathcal{T}$ and $a \neq b$. Then by 16, $a \not\equiv b$. Since the union of distinct functions is not a function, we have that $\ulcorner a \urcorner \cup \ulcorner b \urcorner \neq \ulcorner b \urcorner$, which by (21) means that $a \sqcup b \neq b$. Hence $a \not\sqsubseteq b$. ■

We now verify that $\mathcal{T}$ consists of quantale atoms.

LEMMA 2.16. *Let $\mathfrak{Q}$ be an F-ODA or an R-ODA. For any $a \in \mathcal{T}$, a is not the bottom element $\mathbb{O}$ of the quantale, and if $x \sqsubseteq a$ and $x \neq \mathbb{O}$, then $x = a$.*

PROOF. Let $(Q, \bigsqcup, \cdot, \sim)$ be an R-ODA (the case where it is an F-ODA is simpler). Let $a \in \mathcal{T}$. Since $\mathcal{P}$ contains multiple elements, there must be a $b \in \mathcal{T}$ such that $a \neq b$. By Lemma 2.15, $a \not\sqsubseteq b$. This means that a cannot be the bottom element of the quantale.

Now suppose that $x \in Q$ and $x \sqsubseteq a$. We wish to show that either x is the bottom element of the quantale or that $x = a$. By Lemma 2.13, $x = \bigsqcup A$ for some (possibly empty) set $A \subseteq \mathcal{T}$. If $A = \emptyset$, then x is the bottom element. Otherwise, let $b \in A$, and note that by definition of join, $b \sqsubseteq x$. Then since $x \sqsubseteq a$, we have that $b \sqsubseteq a$. Hence by Lemma 2.15, $a = b$. Thus $x = a$. ■

Lemma 2.16 shows that $\mathcal{T}$ is a set of quantale atoms. Lemma 2.13 then ensures that $\mathcal{T}$ contains all atoms.

2.2.3. Units Here we show that both F-ODAs and R-ODAs have units. Let $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ be an orthomodular dynamic algebra (either relation-based or function-based). By definition $(\mathcal{P}, \preceq, \sim)$ is a complete orthomodular lattice. For the duration of this subsection, we denote $\bigvee \mathcal{P}$ by $\mathbb{1}$. Then $\mathbb{1}$ is the top element of $\mathcal{P}$, as $\mathbb{1} \in \mathcal{P}$ by completeness and obviously $p \preceq \mathbb{1}$ for each $p \in \mathcal{P}$.

We will prove that the top element of the complete orthomodular lattice $\mathcal{P}$ is the unit of the orthomodular dynamic algebra $\mathfrak{Q}$:

THEOREM 2.17. $\mathbb{1} \cdot x = x \cdot \mathbb{1} = x$, *for each* $x \in Q$.

Let $X = \{x \in Q \mid \mathbb{1} \cdot x = x \cdot \mathbb{1} = x\}$.
We prove some lemmas.

LEMMA 2.18. $\mathcal{P} \subseteq X$.

PROOF. Let $q \in \mathcal{P}$ be arbitrary. We have, for any $p \in \mathcal{P}$,

$$\begin{aligned}
\ulcorner \mathbb{1} \cdot q \urcorner (p) &= {\sim\sim}((\mathbb{1} \cdot q) \cdot p) \quad \text{(by definition)} \\
&= {\sim\sim}(\mathbb{1} \cdot (q \cdot p)) \quad \text{(by associativity)} \\
&= \ulcorner \mathbb{1} \urcorner (q \cdot p) \quad \text{(by definition)} \\
&= \ulcorner \mathbb{1} \urcorner ({\sim\sim}(q \cdot p)) \quad \text{(by (19))} \\
&= \ulcorner \mathbb{1} \urcorner (f_q(p)) \quad \text{(by(17))} \\
&= f_{\mathbb{1}}(f_q(p)) \quad \text{(by (17))} \\
&= f_q(p) \quad \text{(by } f_q(p) \preceq \mathbb{1} \text{ and orthomodularity))} \\
&= \ulcorner q \urcorner (p) \quad \text{(by (17))}
\end{aligned}$$

$$\begin{aligned}\ulcorner q\cdot \mathbb{1}\urcorner(p) &= \sim\sim((q\cdot \mathbb{1})\cdot p) \quad \text{(by definition)}\\ &= \sim\sim(q\cdot(\mathbb{1}\cdot p)) \quad \text{(by associativity)}\\ &= \ulcorner q\urcorner(\mathbb{1}\cdot p) \quad \text{(by definition)}\\ &= \ulcorner q\urcorner(\sim\sim(\mathbb{1}\cdot p)) \quad \text{(by (19))}\\ &= \ulcorner q\urcorner(f_{\mathbb{1}}(p)) \quad \text{(by(17))}\\ &= \ulcorner q\urcorner(p) \quad \text{(by } p \preceq \mathbb{1} \text{ and orthomodularity))}\end{aligned}$$

Hence $\ulcorner \mathbb{1}\cdot q\urcorner = \ulcorner q\cdot \mathbb{1}\urcorner = \ulcorner q\urcorner$, i.e. $\mathbb{1}\cdot q \equiv q\cdot \mathbb{1} \equiv q$. By (16) $\mathbb{1}\cdot q = q\cdot \mathbb{1} = q$. ■

LEMMA 2.19. *$\mathcal{U} \subseteq X$.*

PROOF. Let $a \in \mathcal{U}$ be arbitrary. We have, for any $p \in \mathcal{P}$,

$$\begin{aligned}\ulcorner a\cdot \mathbb{1}\urcorner(p) &= \sim\sim((a\cdot \mathbb{1})\cdot p) \quad \text{(by definition)}\\ &= \sim\sim(a\cdot(\mathbb{1}\cdot p)) \quad \text{(by associativity)}\\ &= \sim\sim(a\cdot p) \quad \text{(by Lemma 2.18)}\\ &= \ulcorner a\urcorner(p) \quad \text{(by definition)}\\ \ulcorner \mathbb{1}\cdot a\urcorner(p) &= \sim\sim((\mathbb{1}\cdot a)\cdot p) \quad \text{(by definition)}\\ &= \sim\sim(\mathbb{1}\cdot(a\cdot p)) \quad \text{(by associativity)}\\ &= \ulcorner \mathbb{1}\urcorner(a\cdot p) \quad \text{(by definition)}\\ &= \ulcorner \mathbb{1}\urcorner(\sim\sim(a\cdot p)) \quad \text{(by (19))}\\ &= \ulcorner \mathbb{1}\urcorner(\ulcorner a\urcorner(p)) \quad \text{(by definition)}\\ &= f_{\mathbb{1}}(\ulcorner a\urcorner(p)) \quad (\ulcorner a\urcorner(p) \in \mathcal{P} \quad \text{and(17))}\\ &= \ulcorner a\urcorner(p)\end{aligned}$$

Hence $\ulcorner \mathbb{1}\cdot a\urcorner = \ulcorner a\cdot \mathbb{1}\urcorner = \ulcorner a\urcorner$, i.e. $\mathbb{1}\cdot a \equiv a\cdot \mathbb{1} \equiv a$. By (16) $\mathbb{1}\cdot a = a\cdot \mathbb{1} = a$. ■

LEMMA 2.20. *X is closed under $\cdot$.*

PROOF. Let $x, y \in X$ be arbitrary.

$$\begin{aligned}(x\cdot y)\cdot \mathbb{1} &= x\cdot(y\cdot \mathbb{1})\\ &= x\cdot y\\ &= (\mathbb{1}\cdot x)\cdot y\\ &= \mathbb{1}\cdot(x\cdot y)\end{aligned}$$

■

LEMMA 2.21. *X is closed under $\bigsqcup$.*

Proof. Let $\mathcal{X} \subseteq X$ be arbitrary.

$$\begin{aligned} \mathbb{1} \cdot \bigsqcup \mathcal{X} &= \bigsqcup \{\mathbb{1} \cdot x \mid x \in \mathcal{X}\} \\ &= \bigsqcup \{x \mid x \in \mathcal{X}\} \\ &= \bigsqcup \mathcal{X} \\ &= \bigsqcup \{x \mid x \in \mathcal{X}\} \\ &= \bigsqcup \{x \cdot \mathbb{1} \mid x \in \mathcal{X}\} \\ &= (\bigsqcup \mathcal{X}) \cdot \mathbb{1} \end{aligned}$$

■

Finally, we prove Theorem 2.17.

Proof of Theorem 2.17. By Lemmas 2.18 to 2.21 and (15) $X = Q$. ■

3. Building ODAs from COLs

In this section, we construct from a complete orthomodular lattice $\mathfrak{L}$ an R-ODA denoted by $\mathbf{F}^R(\mathfrak{L})$ and an F-ODA denoted by $\mathbf{F}^F(\mathfrak{L})$. When we need not distinguish between the type of ODA, we may write $\star$-ODA and $\mathbf{F}^\star$ for $\star \in \{F, R\}$. Fix a complete orthomodular lattice $\mathfrak{L} = (L, \leq, -^\perp)$, and denote its top element $\mathbb{1}$. Let

23. $\mathcal{A}$ be the set of ortholattice automorphisms on $\mathfrak{L}$;
24. $\mathcal{F}$ be the smallest set containing $\{f_p \mid p \in L\} \cup \mathcal{A}$, and that is closed under function composition $\circ$ (recall that f_p is the Sasaki projection onto p);
25. $\mathcal{Q}^R \overset{\text{def}}{=} \{\bigcup A \mid A \in \wp(\mathcal{F})\}$ (a set of relations);
26. $\mathcal{Q}^F \overset{\text{def}}{=} \wp(\mathcal{F})$ (a set of sets of functions);
27. $\overset{R}{\cdot} : \mathcal{Q}^R \times \mathcal{Q}^R \to \mathcal{Q}^R :: A \cdot B \mapsto A \circ B$ (relation composition);
28. $\overset{F}{\cdot} : \mathcal{Q}^F \times \mathcal{Q}^F \to \mathcal{Q}^F :: A \cdot B \mapsto \{a \circ b \in \mathcal{F} \mid a \in A \text{ and } b \in B\}$ (pairwise function composition);
29. $\overset{\mathrm{R}}{\sim} : \mathcal{Q}^R \to \mathcal{Q}^R :: A \mapsto f_{(\bigvee\{b \mid (\mathbb{1}, b) \in A\})^\perp}$.
30. $\overset{\mathrm{F}}{\sim} : \mathcal{Q}^F \to \mathcal{Q}^F :: A \mapsto \{f_{(\bigvee\{a(\mathbb{1}) \mid a \in A\})^\perp}\}$.

It is easy to verify that for $\star \in \{R, F\}$, we indeed have that $A \overset{\star}{\cdot} B \in \mathcal{Q}^\star$ and $\overset{\star}{\sim} A \in \mathcal{Q}^\star$ as required by the specified range. We furthermore observe that $\mathcal{Q}^\star$ is closed under arbitrary unions. Given any collection of $X_i \in \mathcal{Q}^R$, each $X_i = \bigcup A_i$ for some $A_i \in \wp(\mathcal{F})$. Hence $\bigcup_i X_i = \bigcup \bigcup_i A_i$, which is in $\mathcal{Q}^R$, since $\bigcup_i A_i \in \wp(\mathcal{F})$. Meanwhile, $\mathcal{Q}^F$ is closed under arbitrary unions, since it consists of a power set. Thus for $\star \in \{R, F\}$, $\mathbf{F}^\star(\mathfrak{L}) \overset{\text{def}}{=} (\mathcal{Q}^\star, \bigcup, \overset{\star}{\cdot}, \overset{\star}{\sim})$ is a generalized dynamic algebra, where the arbitrary set union operator $\bigcup$ plays the role of $\bigsqcup$. We will show that $\mathbf{F}^\star(\mathfrak{L})$ is a $\star$-ODA by verifying the conditions in the definition one by one.

We use the notation defined immediately below Definition 2.8, but often with superscripts indicating whether they are induced from $\mathbf{F}^R(\mathfrak{L})$ or $\mathbf{F}^F(\mathfrak{L})$, in particular for $\star \in \{F, R\}$, we use the corresponding notation $\mathcal{P}^\star$, $\mathcal{U}^\star$, $\mathcal{T}^\star$, $\bigvee^\star$, $\bigwedge^\star$, $\preceq^\star$, and $\ulcorner A \urcorner^\star$. Since the join operator $\overset{\star}{\bigsqcup}$ is set union $\bigcup$, its induced relation $\sqsubseteq^\star$ is just the subset relation $\subseteq$. We may drop superscripts when they should be understood by context. Moreover, here $\mathcal{T}^R$ is $\mathcal{F}$, and $\mathcal{T}^F$ is $\{\{a\} \mid a \in \mathcal{F}\}$.

We now observe that for $\star \in \{F, R\}$, $(\mathcal{Q}^\star, \sqsubseteq^\star, \overset{\star}{\cdot})$ is a quantale. Toward this, recall that the relation $\sqsubseteq^\star$ is simply set containment, and since $\mathcal{Q}^\star$ is closed under arbitrary joins, $(\mathcal{Q}^\star, \sqsubseteq^\star)$ is a complete lattice. Furthermore, since function and relation compositions $\circ$ are both associative, it is easy to see that $\overset{\star}{\cdot}$ is associative. Moreover, the distributivity between $\bigcup$ and $\overset{\star}{\cdot}$ is easy to show. This leads to the following generalization of [5, Proposition 1].

Proposition 3.1. *For both $\star \in \{F, R\}$, $\mathbf{F}^\star(\mathfrak{L})$ satisfies (12), i.e. $(\mathcal{Q}^\star, \sqsubseteq^\star, \overset{\star}{\cdot})$ is a quantale.*

We next provide a useful calculation that extends [5, Lemma 4] to account for R-ODAs as well. The F-ODA component of the proof is similar to that for [5, Lemma 4], but we include it here for completeness.

Lemma 3.2. *For both $\star \in \{R, F\}$ and any $A \in \mathcal{Q}^\star$,ss*

$$\overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim} A = f_{\bigvee\{y \mid (\mathbb{1}, y) \in A\}} \qquad \text{if} \star = R$$

$$\overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim} A = \{f_{\bigvee\{a(\mathbb{1}) \mid a \in A\}}\} \qquad \text{if} \star = F$$

Proof.

$$\overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim} A = \overset{\mathrm{R}}{\sim} f_{(\bigvee\{y \mid (\mathbb{1}, y) \in A\})^\perp} = f_{(f_{(\bigvee\{y \mid (\mathbb{1}, y) \in A\})^\perp}(\mathbb{1}))^\perp} = f_{(\bigvee\{y \mid (\mathbb{1}, y) \in A\})^{\perp\perp}}$$
$$= f_{\bigvee\{y \mid (\mathbb{1}, y) \in A\}}.$$

$$\overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim}A = \overset{\mathrm{F}}{\sim}\{f_{(\bigvee\{a(\mathbb{1})|a\in A\})^{\perp}}\} = \{f_{(f_{(\bigvee\{a(\mathbb{1})|a\in A\})^{\perp}}(\mathbb{1}))^{\perp}}\} = \{f_{(\bigvee\{a(\mathbb{1})|a\in A\})^{\perp\perp}}\}$$
$$= \{f_{\bigvee\{a(\mathbb{1})|a\in A\}}\}.$$

■

A useful special case, involving functions $A \in \mathcal{F}$ (the smallest set which contains $\{f_p \mid p \in L\}$ and $\mathcal{A}$ and is closed under the function composition $\circ$), is as follows:

LEMMA 3.3. *For $A \in \mathcal{F}$,*

$$\ulcorner A \urcorner^{R}(f_p) = f_{A(p)} \qquad \textit{and} \qquad \ulcorner\{A\}\urcorner^{F}(\{f_p\}) = \{f_{A(p)}\}.$$

PROOF. Note that A is a function. Then by Lemma 3.2: $\ulcorner A\urcorner^{R}(f_p) = \overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim}(A\circ f_p) = f_{\bigvee\{y|y\in(A\circ f_p)(\mathbb{1})\}} = f_{(A\circ f_p)(\mathbb{1})} = f_{A(p)}$.

$\ulcorner\{A\}\urcorner^{F}(\{f_p\}) = \overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim}(\{A\circ f_p\}) = \{f_{(A\circ f_p)(\mathbb{1})}\} = \{f_{A(p)}\}$. ■

To clarify the relationships among $\mathcal{P}^F$, $\mathcal{P}^R$, and the set $\{f_p \mid p \in L\}$, we define the following maps:

$$\chi^R : L \to \mathcal{Q}^R :: p \to f_p \tag{31}$$
$$\chi^F : L \to \mathcal{Q}^F :: p \to \{f_p\} \tag{32}$$

Similar to [5, Proposition 2], we show that χ^R and χ^F are both isomorphisms.

PROPOSITION 3.4. *For $\star \in \{R, F\}$, $\chi^\star$ is an ortholattice isomorphism from $\mathfrak{L}$ to $(\mathcal{P}^\star, \preceq^\star, \overset{\star}{\sim})$.*

PROOF. First we show injectivity of $\chi^\star$ for both $\star \in \{R, F\}$. Assume that $p, q \in L$ satisfy $\chi^\star(p) = \chi^\star(q)$. Then $f_p = f_q$. By definition $p = f_p(\mathbb{1}) = f_q(\mathbb{1}) = q$.

Second, we show surjectivity. Note that, for each $A \in \mathcal{Q}^R$,

$$\begin{aligned} A \in \mathcal{P}^R &\Leftrightarrow A = \overset{\mathrm{R}}{\sim}B, \text{ for some } B \in \mathcal{Q}^R \\ &\Leftrightarrow A = f_{(\bigvee\{y|(\mathbb{1},y)\in B\})^{\perp}}, \text{ for some } B \in \mathcal{Q}^R \\ &\Leftrightarrow A = f_p, \text{ for some } p \in L \text{for} \Leftarrow, \text{take} B = f_{p^{\perp}} \end{aligned}$$

Therefore, $\mathcal{P}^R = \{f_p \mid p \in L\}$ and the surjectivity of χ^R follows. Similarly for $A \in \mathcal{Q}^F$,

$$\begin{aligned} A \in \mathcal{P}^F &\Leftrightarrow A = \overset{\mathrm{F}}{\sim}B, \text{ for some } B \in \mathcal{Q}^F \\ &\Leftrightarrow A = \{f_{(\bigvee\{b(\mathbb{1})|b\in B\})^{\perp}}\}, \text{ for some } B \in \mathcal{Q}^F \end{aligned}$$

$$\Leftrightarrow A = \{f_p\}, \text{ for some } p \in L \text{for} \Leftarrow, \text{take} B = \{f_{p^\perp}\}$$

Therefore, $\mathcal{P}^F = \{\{f_p\} \mid p \in L\}$ and the surjectivity of χ^Q follows.

Third, we show that χ^R and χ^F preserve the partial order. Note that, for any $P \subseteq L$,

$$\overset{R}{\bigvee}\{\chi^R(p) \mid p \in P\} = \overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim}\bigcup\{\chi^R(p) \mid p \in P\} = \overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim}\bigcup\{f_p \mid p \in P\} = f_{\bigvee P} = \chi^R(\bigvee P)$$

$$\overset{F}{\bigvee}\{\chi^F(p) \mid p \in P\} = \overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim}\bigcup\{\chi^F(p) \mid p \in P\} = \overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim}\{f_p \mid p \in P\} = \{f_{\bigvee\{f_p(\mathbb{1}) \mid p \in P\}}\}$$

$$\chi^F(\bigvee P) = \{f_{\bigvee P}\} = \{f_{\bigvee\{p \mid p \in P\}}\} = \{f_{\bigvee\{f_p(\mathbb{1}) \mid p \in P\}}\}$$

Hence for each $\star \in \{R, F\}$, $\bigvee^\star\{\chi^\star(p) \mid p \in P\} = \chi^\star(\bigvee P)$. It follows that, for any $p, q \in L$,

$$p \leq q \;\Leftrightarrow\; p \vee q = q \;\Leftrightarrow\; \chi^\star(p \vee q) = \chi^\star(q) \;\Leftrightarrow\; \chi^\star(p) \vee \chi^\star(q) = \chi^\star(q)$$
$$\Leftrightarrow \chi^\star(p) \preceq^\star \chi^\star(q)$$

Finally, we show that χ^R and χ^F preserve orthocomplements. For each $p \in L$,

$$\overset{\mathrm{R}}{\sim}\chi^R(p) = \overset{\mathrm{R}}{\sim} f_p = f_{p^\perp} = \chi^R(p^\perp), \text{ and}$$
$$\overset{\mathrm{F}}{\sim}\chi^F(p) = \overset{\mathrm{F}}{\sim}\{f_p\} = \{f_{(f_p(\mathbb{1}))^\perp}\} = \{f_{p^\perp}\} = \chi^F(p^\perp).$$

■

Because $\mathfrak{L}$ is a complete orthomodular lattice, and $\chi^\star$ is an ortholattice isomorphism thus preserving the property of being a complete orthomodular lattice, we have the following corollary to Proposition 3.4:

Corollary 3.5. *For* $\star \in \{R, F\}$, $\mathbf{F}^\star(\mathfrak{L})$ *satisfies (13), that is* $(\mathcal{P}^\star, \preceq^\star, \overset{\star}{\sim})$ *is a complete orthomodular lattice.*

The exact relationship between $\mathcal{P}^\star$ and $\{f_p \mid p \in L\}$ was shown in the proof of Proposition 3.4, but it also follows as a corollary to that proposition, as $\mathcal{P}^\star$ is the image of $\chi^\star$:

Corollary 3.6. $\mathcal{P}^R = \{f_p \mid p \in L\}$ *and* $\mathcal{P}^F = \{\{f_p\} \mid p \in L\}$.

The following shows that $\mathcal{A}$ is related to $\mathcal{U}^F$ and $\mathcal{U}^R$ as desired.

Proposition 3.7. $\mathcal{U}^R = \mathcal{A}$ *and* $\mathcal{U}^F = \{\{a\} \mid a \in \mathcal{A}\}$.

Proof. Let $A \in \mathcal{A}$, and let $A^R = A$ and $A^F = \{A\}$. Since $A^{-1} \in \mathcal{A}$, A^R and A^F are both invertible. For both $\star \in \{F, R\}$ and $P \in \mathcal{P}^\star$, it is immediate from Lemma (3.3) and the definition of $\chi^\star$ that

$$\ulcorner A^\star \urcorner^\star = \chi^\star \circ A \circ (\chi^\star)^{-1}.$$

Hence $\ulcorner A^\star \urcorner^\star$ is a composition of ortholattice isomorphisms, and is thus an isomorphism on $(\mathcal{P}^\star, \preceq^\star, \overset{\star}{\sim})$. Thus $A \in \mathcal{U}^F$ and $\{A\} \in \mathcal{U}^R$.

Conversely, assume $A \in \mathcal{U}^R$. Then $B \overset{\text{def}}{=} (\chi^R)^{-1} \circ \ulcorner A \urcorner^R \circ \chi^R$ is a composition of ortholattice isomorphisms, and is hence in $\mathcal{A}$. By Lemma 3.3, for each $f_p \in \mathcal{P}^R$, $\ulcorner B \urcorner^R(f_p) = f_{B(p)} = (\chi^R \circ B \circ (\chi^R)^{-1})(f_p) = \ulcorner A \urcorner^R(f_p)$. Thus $A \equiv^R B$. Since A is invertible, it must be a function. Then for any $p \in L$,

$$\begin{aligned}\ulcorner A \urcorner^R(f_p) = \ulcorner B \urcorner^R(f_p) &\Leftrightarrow \overset{\text{R}}{\sim}\overset{\text{R}}{\sim}(A \overset{R}{\cdot} f_p) = \overset{\text{R}}{\sim}\overset{\text{R}}{\sim}(B \overset{R}{\cdot} f_p)\\ &\Leftrightarrow f_{\bigvee\{y \mid y=(A\circ f_p)(\mathbb{1})\}} = f_{\bigvee\{y \mid y=(B\circ f_p)(\mathbb{1})\}} (\text{Lemma}3.2)\\ &\Leftrightarrow f_{A(p)} = f_{B(p)}\\ &\Leftrightarrow A(p) = B(p)\end{aligned}$$

Hence $A = B$. As $B \in \mathcal{A}$ this means $A \in \mathcal{A}$ as desired.

Similarly if $\{A\} \in \mathcal{U}^F$, then $B \overset{\text{def}}{=} (\chi^F)^{-1} \circ \ulcorner \{A\} \urcorner^F \circ \chi^F$ is a composition of ortholattice isomorphisms, and is hence in $\mathcal{A}$. By Lemma 3.3, for each $\{f_p\} \in \mathcal{P}^F$, $\ulcorner B \urcorner^F(\{f_p\}) = \{f_{B(p)}\} = \chi^F \circ B \circ (\chi^F)^{-1}(\{f_p\}) = \ulcorner A \urcorner^F(\{f_p\})$. Thus $A \equiv^F B$. Since A is invertible, it must be a function. Then for any $p \in L$,

$$\begin{aligned}\ulcorner A \urcorner^F(\{f_p\}) = \ulcorner B \urcorner^F(\{f_p\}) &\Leftrightarrow \overset{\text{F}}{\sim}\overset{\text{F}}{\sim}(A \overset{F}{\cdot} \{f_p\}) = \overset{\text{F}}{\sim}\overset{\text{F}}{\sim}(B \overset{F}{\cdot} \{f_p\})\\ &\Leftrightarrow \overset{\text{F}}{\sim}\overset{\text{F}}{\sim}(\{A \circ f_p\}) = \overset{\text{F}}{\sim}\overset{\text{F}}{\sim}(\{B \circ f_p\})\\ &\Leftrightarrow \{f_{(A\circ f_p)(\mathbb{1})}\} = \{f_{(B\circ f_p)(\mathbb{1})}\} \quad (\text{Lemma } 3.2)\\ &\Leftrightarrow A(p) = B(p)\end{aligned}$$

Hence $A = B$. As $B \in \mathcal{A}$ this means $A \in \mathcal{A}$ as desired. ■

PROPOSITION 3.8. *For both $\star \in \{R, F\}$, $\mathbf{F}^\star(\mathfrak{L})$ satisfies (14), that is, for any ortholattice automorphism $f : \mathcal{P}^\star \to \mathcal{P}^\star$, there is an $A^\star \in \mathcal{U}^\star$ such that $\ulcorner A^\star \urcorner^\star = f$.*

PROOF. Let $f : \mathcal{P}^\star \to \mathcal{P}^\star$ be an ortholattice automorphism. Then $A = (\chi^\star)^{-1} \circ f \circ \chi^\star : L \to L$ is the composition of ortholattice isomorphisms and is hence an ortholattice automorphism, that is $A \in \mathcal{A}$. Let $A^R = A$ and $A^F = \{A\}$. By Proposition 3.7, $A^\star \in \mathcal{U}^\star$. From Lemma 3.3 and the definition of $\chi^\star$, it is easy to see that $\ulcorner A^\star \urcorner^\star = \chi^\star \circ A \circ (\chi^\star)^{-1} = f$ as desired. ■

Given the connection between $\{f_p \mid p \in L\}$ and $\mathcal{P}^\star$ (Corollary 3.6), and between $\mathcal{A}$ and $\mathcal{U}^\star$ (Proposition 3.7), it is immediate from the definition of $\mathcal{Q}^\star$ that $\mathcal{Q}^\star$ is appropriately minimal, that is, we have the following analog of [5, Proposition 3]:

PROPOSITION 3.9. *For both $\star \in \{R, F\}$, $\mathbf{F}^\star(\mathfrak{L})$ satisfies (15), i.e. $\mathcal{Q}^\star$ is the smallest set containing $\mathcal{P}^\star$ and $\mathcal{U}^\star$ that is closed under $\overset{\star}{\cdot}$ and $\bigcup$.*

The F-ODA component of the following proposition is similar to [5, Proposition 5], but we include its proof here for completeness.

PROPOSITION 3.10. *For both $\star \in \{R, F\}$, $\mathbf{F}^\star(\mathfrak{L})$ satisfies (16), i.e. for any $A, B \in \mathcal{T}^\star$, the following are equivalent:*

(i) $A = B$;

(ii) $A \equiv^\star B$, *i.e.* $\ulcorner A \urcorner^\star(P) = \ulcorner B \urcorner^\star(P)$ *for each* $P \in \mathcal{P}^\star$.

PROOF. The direction from (i) to (ii) is obvious, so we focus on the direction from (ii) to (i). For the case where $\star = R$, since $A, B \in \mathcal{T}^R$, by definition there are $m, n \in \mathbb{N}^+$ and $g_1, g_2, \dots g_m, h_1, \dots, h_n \in \{f_p \mid p \in L\} \cup \mathcal{A}$, such that $A = g_1 \circ \cdots \circ g_m$ and $B = h_1 \circ \cdots \circ h_n$. By Corollary 3.6, for each $p \in L$, $f_p \in \mathcal{P}^R$ and

$$\begin{aligned}
\ulcorner A \urcorner^R(f_p) = \ulcorner B \urcorner^R(f_p) &\Leftrightarrow \overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim}(A \overset{R}{\cdot} f_p) = \overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim}(B \overset{R}{\cdot} f_p)\\
&\Leftrightarrow \overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim}((g_1 \circ \cdots \circ g_m) \overset{R}{\cdot} f_p) = \overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim}((h_1 \circ \cdots \circ h_n) \overset{R}{\cdot} (f_p))\\
&\Leftrightarrow \overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim}(g_1 \circ \cdots \circ g_m \circ f_p) = \overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim}(h_1 \circ \cdots \circ h_n \circ f_p)\\
&\Leftrightarrow f_{\bigvee\{y \mid y = (g_1 \circ \cdots \circ g_m \circ f_p)(\mathbb{1})\}} = f_{\bigvee\{y \mid y = (h_1 \circ \cdots \circ h_n \circ f_p)(\mathbb{1})\}} \quad \text{(Lemma 3.2)}\\
&\Leftrightarrow f_{(g_1 \circ \cdots \circ g_m \circ f_p)(\mathbb{1})} = f_{(h_1 \circ \cdots \circ h_n \circ f_p)(\mathbb{1})}\\
&\Leftrightarrow (g_1 \circ \cdots \circ g_m)(p) = (h_1 \circ \cdots \circ h_n)(p)
\end{aligned}$$

Now for each $p \in L$, by (ii) and the above equivalence we have $(g_1 \circ \cdots \circ g_m)(p) = (h_1 \circ \cdots \circ h_n)(p)$. Hence $g_1 \circ \cdots \circ g_m = h_1 \circ \cdots \circ h_n$. Therefore, $A = B$.

For the case where $\star = F$, since $A, B \in \mathcal{T}^F$, by definition there are $m, n \in \mathbb{N}^+$ and $g_1, g_2, \dots g_m, h_1, \dots, h_n \in \{f_p \mid p \in L\} \cup \mathcal{A}$, such that $A = \{g_1 \circ \cdots \circ g_m\}$ and $B = \{h_1 \circ \cdots \circ h_n\}$. Then by Corollary 3.6, for each $p \in L$, $\{f_p\} \in \mathcal{P}^F$ and

$$\begin{aligned}
\ulcorner A \urcorner^F(\{f_p\}) = \ulcorner B \urcorner^F(\{f_p\}) &\Leftrightarrow \overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim}(A \overset{F}{\cdot} \{f_p\}) = \overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim}(B \overset{F}{\cdot} \{f_p\})\\
&\Leftrightarrow \overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim}(\{g_1 \circ \cdots \circ g_m\} \overset{F}{\cdot} \{f_p\}) = \overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim}(\{h_1 \circ \cdots \circ h_n\} \overset{F}{\cdot} \{f_p\})\\
&\Leftrightarrow \overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim}(\{g_1 \circ \cdots \circ g_m \circ f_p\}) = \overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim}(\{h_1 \circ \cdots \circ h_n \circ f_p\})\\
&\Leftrightarrow \{f_{(g_1 \circ \cdots \circ g_m \circ f_p)(\mathbb{1})}\} = \{f_{(h_1 \circ \cdots \circ h_n \circ f_p)(\mathbb{1})}\} \quad \text{(Lemma 3.2)}\\
&\Leftrightarrow (g_1 \circ \cdots \circ g_m)(p) = (h_1 \circ \cdots \circ h_n)(p)
\end{aligned}$$

Now for each $p \in L$, by (ii) and the above equivalence we have $(g_1 \circ \cdots \circ g_m)(p) = (h_1 \circ \cdots \circ h_n)(p)$. Hence $g_1 \circ \cdots \circ g_m = h_1 \circ \cdots \circ h_n$. Therefore, $A = B$. ∎

The F-ODA component of the following proposition is similar to [5, Proposition 6], but we include its proof here for completeness.

Proposition 3.11. *For each* $\star \in \{R, F\}$, $\mathbf{F}^\star(\mathfrak{L})$ *satisfies (17): for any* $A, B \in \mathcal{P}^\star$, $\ulcorner A \urcorner^\star(B) = f_A(B)$.

Proof. Let $\star \in \{R, F\}$. By Proposition 3.4 there are $p, q \in L$ such that

$$A = \begin{cases} \chi^R(p) = f_p & \text{if} \star = R \\ \chi^F(p) = \{f_p\} & \text{if} \star = F \end{cases}, \qquad B = \begin{cases} \chi^R(q) = f_q & \text{if} \star = R \\ \chi^F(q) = \{f_q\} & \text{if} \star = F \end{cases}.$$

Then by Lemma 3.3 and Proposition 3.4: if $\star = R$, then

$$\begin{aligned} \ulcorner A \urcorner^R(B) &= \ulcorner f_p \urcorner^R(f_q) = f_{f_p(q)} = \chi^R(p \wedge (p^\perp \vee q)) \\ &= \chi^R(p) \wedge (\overset{\mathrm{R}}{\sim} \chi^R(p) \vee \chi^R(q)) = A \wedge (\overset{\mathrm{R}}{\sim} A \vee B) = f_A(B). \end{aligned}$$

if $\star = F$, then

$$\begin{aligned} \ulcorner A \urcorner^F(B) &= \ulcorner \{f_p\} \urcorner^F(\{f_q\}) = \{f_{f_p(q)}\} = \chi^F(p \wedge (p^\perp \vee q)) \\ &= \chi^F(p) \wedge (\overset{\mathrm{F}}{\sim} \chi^F(p) \vee \chi^F(q)) = A \wedge (\overset{\mathrm{F}}{\sim} A \vee B) = f_A(B). \end{aligned}$$

Therefore for both $\star \in \{F, R\}$, $\ulcorner A \urcorner^\star(B) = f_A(B)$. ■

The following proposition concerns unitaries, and effectively states that, for $A \in \mathcal{U}^\star$ and $P \in \mathcal{P}^\star$, $\ulcorner A \urcorner^\star(P)$ is the conjugate of P by A.

Proposition 3.12. *For* $\star \in \{R, F\}$, $\mathbf{F}^\star(\mathfrak{L})$ *satisfies (18), that is, for* $A \in \mathcal{U}^\star$ *and* $P \in \mathcal{P}^\star$, $A \overset{\star}{\cdot} P = \ulcorner A \urcorner^\star(P) \overset{\star}{\cdot} A$.

Proof. If $\star = R$, then $A \in \mathcal{A}$ and there is a $p \in L$ such that $P = f_p$. Then by Lemma 2.4 $A \overset{R}{\cdot} P = A \circ f_p = f_{A(p)} \circ A = \ulcorner A \urcorner^R(P) \overset{R}{\cdot} A$ as desired.

If $\star = F$, then there is a $B \in \mathcal{A}$ such that $A = \{B\}$, and there is a $p \in L$ such that $P = \{f_p\}$. Thus by Lemma 2.4 $A \overset{F}{\cdot} P = \{B \circ f_p\} = \{f_{B(p)} \circ B\} = \ulcorner A \urcorner^F(P) \overset{F}{\cdot} A$ as desired. ■

The F-ODA component of the following proposition is similar to [5, Proposition 7], but we include its proof here for completeness.

Proposition 3.13. *For each* $\star \in \{R, F\}$, $\mathbf{F}^\star(\mathfrak{L})$ *satisfies (19), i.e. for any* $A, B \in \mathcal{Q}^\star$, $\ulcorner A \urcorner^\star(B) = \ulcorner A \urcorner^\star(\overset{\star}{\sim} \overset{\star}{\sim} B)$.

Proof. Recall that by Lemma 2.2 Sasaki projections preserve arbitrary joins, and thus the same holds for compositions of Sasaki projections and ortholattice isomorphisms.

For $\mathbf{F}^R(\mathfrak{L})$, suppose $A = \bigcup\{g_i \mid i \in I\}$ and $B = \bigcup\{h_j \mid j \in J\}$ where each g_i and h_j are compositions of Sasaki projections and ortholattice isomorphisms, and hence preserve arbitrary joins. Then $\ulcorner A \urcorner^R(\overset{R}{\sim}\overset{R}{\sim} B) = \overset{R}{\sim}\overset{R}{\sim}(A \cdot \overset{R}{\sim}\overset{R}{\sim} B) = \overset{R}{\sim}\overset{R}{\sim}(A \cdot f_{\bigvee\{b|(\mathbb{1},b)\in B\}}) = \overset{R}{\sim}\overset{R}{\sim}(A \cdot f_{\bigvee_{j\in J})h_j(\mathbb{1})}) = f_{\bigvee\{a|(f_{\bigvee_{j\in J} h_j(\mathbb{1})}(\mathbb{1}),a)\in A\}}$ $= f_{\bigvee\{a|(\bigvee_{j\in J} h_j(\mathbb{1}),a)\in A\}} = f_{\bigvee_{i\in I}\{g_i(\bigvee_{j\in J} h_j(\mathbb{1}))\}} = f_{\bigvee_{i\in I, j\in J} g_i(h_j(\mathbb{1}))} = f_{\bigvee\{c|(\mathbb{1},c)\in A\circ B\}} = \overset{R}{\sim}\overset{R}{\sim}(A \cdot B) = \ulcorner A \urcorner^R(B)$.

For $\mathbf{F}^F(\mathfrak{L})$, $\ulcorner A \urcorner^F(\overset{F}{\sim}\overset{F}{\sim} B) = \overset{F}{\sim}\overset{F}{\sim}(A \cdot \overset{F}{\sim}\overset{F}{\sim} B) = \overset{F}{\sim}\overset{F}{\sim}(A \cdot \{f_{\bigvee\{b(\mathbb{1})|b\in B\}}\}) = \overset{F}{\sim}\overset{F}{\sim}\{a \circ f_{\bigvee_{b\in B} b(\mathbb{1})} \mid a \in A\} = \{f_{\bigvee\{(a\circ f_{\bigvee_{b\in B} b(\mathbb{1})})(\mathbb{1})|a\in A\}}\} = \{f_{\bigvee\{a(\bigvee_{b\in B} b(\mathbb{1}))|a\in A\}}\}$ $= \{f_{\bigvee\{(a\circ b)(\mathbb{1})|a\in A, b\in B\}}\}$.

Since $\ulcorner A \urcorner^F(B) = \overset{F}{\sim}\overset{F}{\sim}(A \cdot B) = \overset{F}{\sim}\overset{F}{\sim}\{a \circ b \mid a \in A, b \in B\}$ $= \{f_{\bigvee\{(a\circ b)(\mathbb{1})|a\in A, b\in B\}}\}$, $\ulcorner A \urcorner^F(\overset{F}{\sim}\overset{F}{\sim} B) = \{f_{\bigvee\{(a\circ b)(\mathbb{1})|a\in A, b\in B\}}\} = \ulcorner A \urcorner^F(B)$. ∎

An interesting consequence of this is that $\ulcorner - \urcorner^R$ is injective: By Proposition 3.13, for any $X, Y \subseteq \mathcal{T}^R$, and any $A \in X$ and $B \in Y$, $\ulcorner A \urcorner = \ulcorner B \urcorner$ iff $\ulcorner A \urcorner \equiv \ulcorner B \urcorner$. Then by Proposition 3.10, the mapping from A to $\ulcorner A \urcorner$ is injective. Hence we have the following:

Corollary 3.14. *For any $X, Y \subseteq \mathcal{T}^R$, it holds that $X = Y$ if and only if $\{\ulcorner x \urcorner \mid x \in X\} = \{\ulcorner y \urcorner \mid y \in Y\}$.*

Since the ODAs of [5] did not involve (20), this following result is new to this work.

Proposition 3.15. *For each $\star \in \{R, F\}$, $\mathbf{F}^\star(\mathfrak{L})$ satisfies (20), i.e. $\overset{\star}{\sim}\bigcup\{A_i \mid i \in I\} = \overset{\star}{\sim}\bigcup\{\overset{\star}{\sim}\overset{\star}{\sim} A_i \mid i \in I\}$, whenever $\{A_i \mid i \in I\} \subseteq \mathcal{T}^\star$.*

Proof. For $\mathbf{F}^R(\mathfrak{L})$,

$$
\begin{aligned}
\overset{R}{\sim}\bigcup\{A_i \mid i \in I\} &= f_{\left(\bigvee\{y|(\mathbb{1},y)\in\bigcup\{A_i|i\in I\}\}\right)^\perp} \\
&= f_{\left(\bigvee\{y|\text{there is an } i\in I \text{ such that } (\mathbb{1},y)\in A_i\}\right)^\perp} \\
&= f_{\left(\bigvee\{y|\text{there is an } i\in I \text{ such that } y=A_i(\mathbb{1})\}\right)^\perp} \\
&\qquad (\text{since} A_i \in \mathcal{T}^R, A_i \text{is a function}) \\
&= f_{\left(\bigvee\{A_i(\mathbb{1})|i\in I\}\right)^\perp} \\
\overset{R}{\sim}\bigcup\{\overset{R}{\sim}\overset{R}{\sim} A_i \mid i \in I\} &= f_{\left(\bigvee\{y|(\mathbb{1},y)\in\bigcup\{\overset{R}{\sim}\overset{R}{\sim} A_i|i\in I\}\}\right)^\perp} \\
&= f_{\left(\bigvee\{y|\text{there is an } i\in I \text{ such that } (\mathbb{1},y)\in\overset{R}{\sim}\overset{R}{\sim} A_i\}\right)^\perp}
\end{aligned}
$$

$$= f_{\left(\bigvee\{y | \text{there is an } i \in I \text{ such that } (\mathbb{1}, y) \in f_{\bigvee\{z | (\mathbb{1}, z) \in A_i\}}\}\right)^{\perp}}$$
$$= f_{\left(\bigvee\{y | \text{there is an } i \in I \text{ such that } y = f_{\bigvee\{z | (\mathbb{1}, z) \in A_i\}}(\mathbb{1})\}\right)^{\perp}}$$
$$= f_{\left(\bigvee\{y | \text{there is an } i \in I \text{ such that } y = f_{A_i(\mathbb{1})}(\mathbb{1})\}\right)^{\perp}}$$
$$(\text{since} A_i \in \mathcal{T}^R, A_i \text{is a function})$$
$$= f_{\left(\bigvee\{y | \text{there is an } i \in I \text{ such that } y = A_i(\mathbb{1})\}\right)^{\perp}}$$
$$= f_{\left(\bigvee\{A_i(\mathbb{1}) | i \in I\}\right)^{\perp}}$$

Therefore, $\overset{\mathrm{R}}{\sim} \bigcup \{A_i \mid i \in I\} = f_{\left(\bigvee\{A_i(\mathbb{1}) | i \in I\}\right)^{\perp}} = \overset{\mathrm{R}}{\sim} \bigcup \{\overset{\mathrm{R}}{\sim}\overset{\mathrm{R}}{\sim} A_i \mid i \in I\}$.

For $\mathbf{F}^F(\mathfrak{L})$, as each $A_i \in \mathcal{T}^F$, $A_i = \{x_i\}$, where x_i is a function on L. So by Lemma 3.2, $\overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim} A_i = \overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim}\{x_i\} = f_{x_i(\mathbb{1})} \in \mathcal{P}^F$. Then

$$\begin{aligned}
\overset{\mathrm{F}}{\sim} \bigcup \{\overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim} A_i \mid i \in I\} &= \left\{ f_{\left(\bigvee\{a(\mathbb{1}) | a \in \bigcup\{\overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim} A_i | i \in I\}\right)^{\perp}} \right\} \\
&= \left\{ f_{\left(\bigvee\{a(\mathbb{1}) | a \in \bigcup\{\{f_{x_i(\mathbb{1})}\} | i \in I\}\right)^{\perp}} \right\} \\
&= \left\{ f_{\left(\bigvee\{f_{x_i(\mathbb{1})}(\mathbb{1}) | i \in I\}\right)^{\perp}} \right\} \\
&= \left\{ f_{\left(\bigvee\{x_i(\mathbb{1}) | i \in I\}\right)^{\perp}} \right\} \\
&= \overset{\mathrm{F}}{\sim}\{x_i \mid i \in I\} \\
&= \overset{\mathrm{F}}{\sim} \bigcup \{A_i \mid i \in I\}
\end{aligned}$$

Therefore, $\overset{\mathrm{F}}{\sim} \bigcup \{A_i \mid i \in I\} = \overset{\mathrm{F}}{\sim} \bigcup \{\overset{\mathrm{F}}{\sim}\overset{\mathrm{F}}{\sim} A_i \mid i \in I\}$. ■

Condition (21) is particular to R-ODAs. As each element of $\mathcal{Q}^R$ is a relation on L, in this context, condition (21) largely amounts to lifting the construction of relations on L to relations on $\mathcal{Q}^R$.

PROPOSITION 3.16. $\mathbf{F}^R(\mathfrak{L})$ *satisfies (21), i.e. for any* $X, Y \subseteq \mathcal{T}^R$, $\bigcup X = \bigcup Y$ *if and only if* $\bigcup\{\ulcorner x \urcorner^R \mid x \in X\} = \bigcup\{\ulcorner y \urcorner^R \mid y \in Y\}$. *(Here,* $\mathcal{T}^R$ *is the smallest subset of* $\mathcal{Q}^R$ *which contains* $\mathcal{P}^R$ *and* $\mathcal{U}^R$ *and is closed under the operation* $\overset{R}{\cdot}$.*)*

PROOF. We drop the superscripts R as they are understood from context. Suppose $\bigcup X = \bigcup Y$, suppose $(\gamma, \delta) \in \bigcup\{\ulcorner x \urcorner \mid x \in X\}$, and let A be the set of atoms below γ. We wish to show that $(\gamma, \delta) \in \bigcup\{\ulcorner y \urcorner \mid y \in Y\}$. Now there must be some $x \in X$ such that $\delta = \ulcorner x \urcorner(\gamma)$, that is $\delta = \sim\sim(x \circ \gamma) = \sim\sim(\bigcup_{a\in A} x \circ a) = f_{\bigvee_{a\in A} x\circ a(\mathbb{1})}$, which by Lemma 2.2 is equal to $f_{x(\bigvee_{a\in A} a(\mathbb{1}))}$. Let $p = \bigvee_{a\in A} a(\mathbb{1})$ and $q = x(p)$. Thus $(p, q) \in \bigcup X$, and hence $(p, q) \in \bigcup Y$. Thus there is some $y \in Y$, such that $q = y(p)$. Then $f_{\bigvee_{a\in A} x\circ a(\mathbb{1})} = f_{\bigvee_{a\in A} y\circ a(\mathbb{1})}$, from which we can see that $\delta = \ulcorner y \urcorner(\gamma)$. Hence $\bigcup\{\ulcorner x \urcorner \mid x \in X\} \subseteq \bigcup\{\ulcorner y \urcorner \mid y \in Y\}$. The proof that $\bigcup\{\ulcorner x \urcorner \mid x \in X\} \supseteq \bigcup\{\ulcorner y \urcorner \mid y \in Y\}$ is completely symmetric. Thus $\bigcup\{\ulcorner x \urcorner \mid x \in X\} = \bigcup\{\ulcorner y \urcorner \mid y \in Y\}$.

Suppose conversely that $\bigcup\{\ulcorner x \urcorner \mid x \in X\} = \bigcup\{\ulcorner y \urcorner \mid y \in Y\}$. Let $(p, q) \in \bigcup X$. We want to show that $(p, q) \in \bigcup Y$. Now, there is some $x \in X$, such that $q = x(p)$. Furthermore, since $\ulcorner x \urcorner$ maps f_p to f_q, $(f_p, f_q) \in \bigcup\{\ulcorner x \urcorner \mid x \in X\} = \bigcup\{\ulcorner y \urcorner \mid y \in Y\}$. Then there exists $y \in Y$, such that $f_q = \ulcorner y \urcorner(f_p)$. Hence $q = y(p)$, and $(p, q) \in \bigcup Y$. ■

The F-ODA version follows from the Axiom of Extensionality of basic set theory. Thus, just as in [5, Proposition 4], we have the following:

PROPOSITION 3.17. *$\mathbf{F}^F(\mathfrak{L})$ satisfies (22), i.e. for any $X, Y \subseteq \mathcal{T}^F$, $\bigcup X = \bigcup Y$, if and only if $X = Y$.*

THEOREM 3.18. *$\mathbf{F}^R(\mathfrak{L})$ is an R-ODA and $\mathbf{F}^F(\mathfrak{L})$ is an F-ODA.*

PROOF. It follows from the definition and the previous propositions. ■

4. Representation Theorem

According to the definition of ODAs, namely (13), we know that, given an R-ODA (or an F-ODA) $\mathfrak{Q}$, there is a complete orthomodular lattice $\mathfrak{L} = (\mathcal{P}, \preceq, \sim)$ inside $\mathfrak{Q}$. Then, using the construction in Section 3, we can build an R-ODA $\mathbf{F}^R(\mathfrak{L})$ (or an F-ODA $\mathbf{F}^F(\mathfrak{L})$, respectively). In this section, we will prove that $\mathfrak{Q}$ is isomorphic to $\mathbf{F}^R(\mathfrak{L})$ (and $\mathbf{F}^F(\mathfrak{L})$, respectively) in the sense of universal algebra; and thus conclude that each of R-ODAs and F-ODAs can be considered as being built from a complete orthomodular lattice. This is a representation theorem for ODAs.

DEFINITION 4.1. Let $\mathfrak{Q}_1 = (Q_1, \bigsqcup_1, \cdot_1, \sim_1)$ to $\mathfrak{Q}_2 = (Q_2, \bigsqcup_2, \cdot_2, \sim_2)$ be both R-ODAs (both F-ODAs). A function $\theta : Q_1 \to Q_2$ is a *universal algebra isomorphism (UA-isomorphism)* if:

33. θ is a bijection, and
34. θ preserves $\sim$, $\bigsqcup$ and $\cdot$, i.e. for any $A_1 \subseteq \mathcal{Q}_1$ and $x_1, y_1 \in \mathcal{Q}_1$, $\sim_2\theta(x_1) = \theta(\sim_1 x_1)$ $\quad \theta(\bigsqcup_1 A_1) = \bigsqcup_2 \theta[A_1]$ $\quad \theta(x_1 \cdot_1 y_1) = \theta(x_1) \cdot_2 \theta(x_2)$

If there is a UA-isomorphism between $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$, we say that $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are *UA-isomorphic*.

Let $\star \in \{R, F\}$. Given a $\star$-ODA $\mathfrak{Q}$, for each $x \in Q$, let

$$\mathcal{T}^{\star}_{\mathfrak{Q}}(x) = \{y \in \mathcal{T}^{\star}_{\mathfrak{Q}} \mid y \sqsubseteq x\}$$

be the set of atoms below x. Then define

$$\mathbf{U}^{\star}(\mathfrak{Q}) = (\mathcal{P}, \preceq, \sim)$$

which is a complete orthomodular lattice according to (13). We now define a function $\eta^{\star}_{\mathfrak{Q}} : \mathfrak{Q} \to \mathbf{F}^{\star}(\mathbf{U}^{\star}(\mathfrak{Q}))$ as follows:

$$\eta^{R}_{\mathfrak{Q}}{::}x \mapsto \bigcup\{\ulcorner y \urcorner \mid y \in \mathcal{T}^{R}_{\mathfrak{Q}}(x)\},$$
$$\eta^{F}_{\mathfrak{Q}}{::}x \mapsto \{\ulcorner y \urcorner \mid y \in \mathcal{T}^{F}_{\mathfrak{Q}}(x)\}.$$

Although the domain of $\ulcorner y \urcorner$ is all of Q, here $\ulcorner y \urcorner$ is viewed as an operator on $\mathcal{P}$. Recall by atomicity (Lemma 2.13) that any $x \in Q$ is the join of the atoms below it. Thus if $\{x_i \mid\in I\}$ is the set of atoms below x, then

$$\eta^{R}_{\mathfrak{Q}} : x = \bigsqcup\{x_i \mid i \in I\} \mapsto \bigcup\{\ulcorner x_i \urcorner \mid i \in I\},$$
$$\eta^{F}_{\mathfrak{Q}} : x = \bigsqcup\{x_i \mid i \in I\} \mapsto \{\ulcorner x_i \urcorner \mid i \in I\}.$$

For an orthomodular dynamic algebra, $\mathfrak{Q}$, let $\mathbb{0}$ and $\mathbb{1}$ be the bottom and top elements of its corresponding orthomodular lattice $\mathbf{U}^{\star}(\mathfrak{Q})$.

THEOREM 4.2. *For both $\star \in \{R, F\}$ and each $\star$-ODA $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$, $\eta^{\star}_{\mathfrak{Q}}$ is a UA-isomorphism from $\mathfrak{Q}$ to $\mathbf{F}^{\star}(\mathbf{U}^{\star}(\mathfrak{Q}))$.*

PROOF. First we show that $\eta^{\star}_{\mathfrak{Q}}$ is injective. Assume that $x, y \in Q$ are such that $\eta^{\star}_{\mathfrak{Q}}(x) = \eta^{\star}_{\mathfrak{Q}}(y)$.

- In the case where $\star = R$, since $\eta^{R}_{\mathfrak{Q}}(x) = \eta^{R}_{\mathfrak{Q}}(y)$, by definition of $\eta^{R}_{\mathfrak{Q}}$, we have that $\bigcup\{\ulcorner z \urcorner \mid z \in \mathcal{T}^{R}_{\mathfrak{Q}}(x)\} = \bigcup\{\ulcorner z \urcorner \mid z \in \mathcal{T}^{R}_{\mathfrak{Q}}(y)\}$. Then by (21) and Lemma 2.13 $x = \bigsqcup \mathcal{T}^{R}_{\mathfrak{Q}}(x) = \bigsqcup \mathcal{T}^{R}_{\mathfrak{Q}}(y) = y$.
- In the case where $\star = F$, let $x_i \in \mathcal{T}^{F}_{\mathfrak{Q}}(x)$. Then $\ulcorner x_i \urcorner \in \eta^{F}_{\mathfrak{Q}}(x)$. By the assumption that $\eta^{F}_{\mathfrak{Q}}(x) = \eta^{F}_{\mathfrak{Q}}(y)$, there exists a $y_j \in \mathcal{T}^{F}_{\mathfrak{Q}}(y)$ such that $\ulcorner x_i \urcorner = \ulcorner y_j \urcorner$, and hence $x_i \equiv y_j$. By (16), $x_i = y_j$. Symmetrically, we can show that for every $y_j \in \mathcal{T}^{F}_{\mathfrak{Q}}(y)$, there is an $x_i \in \mathcal{T}^{F}_{\mathfrak{Q}}(x)$, such that $x_i = y_j$. Thus $\mathcal{T}^{F}_{\mathfrak{Q}}(x) = \mathcal{T}^{F}_{\mathfrak{Q}}(y)$ and so by Lemma 2.13, $x = \bigsqcup \mathcal{T}^{F}_{\mathfrak{Q}}(x) = \bigsqcup \mathcal{T}^{F}_{\mathfrak{Q}}(y) = y$.

Therefor, $\eta^{\star}_{\mathfrak{Q}}$ is injective.

Next, we show that $\eta^{\star}_{\mathfrak{Q}}$ preserves the algebraic operations. It is not hard to show that $\eta^{\star}_{\mathfrak{Q}}$ preserves $\bigsqcup$. To show that $\eta^{\star}_{\mathfrak{Q}}$ preserves $\cdot$, let $x = \bigsqcup\{x_i \mid i \in I\}$

and $y = \bigsqcup\{y_j \mid j \in J\}$ with $x_i, y_j \in \mathcal{T}^\star$. Then

$$\begin{aligned}
&\eta^\star_{\mathfrak{Q}}(\bigsqcup\{x_i \mid i \in I\} \cdot \bigsqcup\{y_j \mid j \in J\}) \\
&\quad = \eta^\star_{\mathfrak{Q}}(\bigsqcup\{x_i \cdot y_j \mid i \in I, j \in J\}) \quad (\text{by}(11)) \\
&\quad = \bigcup\{\eta^\star_{\mathfrak{Q}}(x_i \cdot y_j) \mid i \in I, j \in J\} \quad (\eta^\star_{\mathfrak{Q}} \text{ preserves} \bigsqcup) \\
&\quad = \bigcup\{\eta^\star_{\mathfrak{Q}}(x_i) \cdot \eta^\star_{\mathfrak{Q}}(y_j) \mid i \in I, j \in J\} \quad (\text{by the definition of } \eta^\star_{\mathfrak{Q}}) \\
&\quad = \bigcup\{\bigcup\{\eta^\star_{\mathfrak{Q}}(x_i) \mid i \in I\} \cdot \eta^\star_{\mathfrak{Q}}(y_j) \mid j \in J\} \\
&\quad = \bigcup\{\eta^\star_{\mathfrak{Q}}(x_i) \mid i \in I\} \cdot \bigcup\{\eta^\star_{\mathfrak{Q}}(y_j) \mid j \in J\}
\end{aligned}$$

Finally we show that $\eta^\star_{\mathfrak{Q}}$ preserves orthocomplementation.

$$\begin{aligned}
\overset{\mathrm{F}}{\sim}\eta^F_{\mathfrak{Q}}(\bigsqcup\{x_i \mid i \in I\}) &= \overset{\mathrm{F}}{\sim}\{\ulcorner x_i \urcorner \mid i \in I\} \\
&= \{f_{\sim \bigvee\{\ulcorner x_i \urcorner(\mathbb{1}) \mid i \in I\}}\} \\
&= \{\ulcorner \sim \bigvee\{\ulcorner x_i \urcorner(\mathbb{1}) \mid i \in I\} \urcorner\} \quad (\text{by } (17)) \\
&= \{\ulcorner \sim\sim\sim \bigsqcup\{\ulcorner x_i \urcorner(\mathbb{1}) \mid i \in I\} \urcorner\} \\
&= \{\ulcorner \sim\sim\sim \bigsqcup\{\sim\sim x_i \mid i \in I\} \urcorner\} \quad (\text{by Theorem } 2.17) \\
&= \{\ulcorner \sim \bigsqcup\{\sim\sim x_i \mid i \in I\} \urcorner\} \\
&= \{\ulcorner \sim \bigsqcup\{x_i \mid i \in I\} \urcorner\} \quad (\text{by } (20)) \\
&= \eta^F_{\mathfrak{Q}}(\sim \bigsqcup\{x_i \mid i \in I\}) \\
\overset{\mathrm{R}}{\sim}\eta^R_{\mathfrak{Q}}(\bigsqcup\{x_i \mid i \in I\}) &= \overset{\mathrm{R}}{\sim}\bigcup\{\ulcorner x_i \urcorner \mid i \in I\} \\
&= f_{\sim \bigvee\{\ulcorner x_i \urcorner(\mathbb{1}) \mid i \in I\}} \quad (\text{since } \ulcorner x_i \urcorner \text{ is a function}) \\
&= \ulcorner \sim \bigvee\{\ulcorner x_i \urcorner(\mathbb{1}) \mid i \in I\} \urcorner \quad (\text{by } (17)) \\
&= \ulcorner \sim\sim\sim \bigsqcup\{\ulcorner x_i \urcorner(\mathbb{1}) \mid i \in I\} \urcorner \\
&= \ulcorner \sim\sim\sim \bigsqcup\{\sim\sim x_i \mid i \in I\} \urcorner \quad (\text{by Theorem } 2.17) \\
&= \ulcorner \sim \bigsqcup\{\sim\sim x_i \mid i \in I\} \urcorner \\
&= \ulcorner \sim \bigsqcup\{x_i \mid i \in I\} \urcorner \quad (\text{by } (20)) \\
&= \eta^R_{\mathfrak{Q}}(\sim \bigsqcup\{x_i \mid i \in I\})
\end{aligned}$$

Finally, we show that $\eta^\star_{\mathfrak{Q}}$ is surjective. We do so by appealing to minimality (15). We first show that all $\mathcal{P}^\star_{\mathbf{F}^\star(\mathbf{U}^\star(\mathfrak{Q}))}$ and $\mathcal{U}^\star_{\mathbf{F}^\star(\mathbf{U}^\star(\mathfrak{Q}))}$ are in the image of $\eta^\star_{\mathfrak{Q}}$; in particular, for any $X \in \mathcal{P}^\star_{\mathbf{F}^\star(\mathbf{U}^\star(\mathfrak{Q}))}$ and any $Y \in \mathcal{U}^\star_{\mathbf{F}^\star(\mathbf{U}^\star(\mathfrak{Q}))}$, there is a $q \in \mathcal{P}_{\mathfrak{Q}}$ and $a \in \mathcal{U}^{\star}_{\mathfrak{Q}}$ such that

$$X = \begin{cases} \ulcorner q \urcorner & \text{if} \star = R \\ \{\ulcorner q \urcorner\} & \text{if} \star = F \end{cases} \quad \text{and} \quad Y = \begin{cases} \ulcorner a \urcorner & \text{if} \star = R \\ \{\ulcorner a \urcorner\} & \text{if} \star = F \end{cases}.$$

- Let $X \in \mathcal{P}^{\star}_{\mathbf{F}^{\star}(\mathbf{U}^{\star}(\mathfrak{Q}))}$. If $\star = R$, then $X = f_q$ for some $q \in \mathcal{P}_{\mathfrak{Q}}$. By (17), $f_q = \ulcorner q \urcorner$, that is $X = \ulcorner q \urcorner$. Similarly if $\star = F$, then $X = \{f_q\}$ for some $q \in \mathcal{P}_{\mathfrak{Q}}$. By (17), $f_q = \ulcorner q \urcorner$, that is $X = \{\ulcorner q \urcorner\}$.
- Let $Y \in \mathcal{U}^{\star}_{\mathbf{F}^{\star}(\mathbf{U}^{\star}(\mathfrak{Q}))}$. Then for some ortholattice automorphism $A \in \mathcal{A}$ on $\mathcal{P}$ we have that if $\star = R$, then $Y = A$ and if $\star = F$, then $Y = \{A\}$. By (14), there is an $a \in \mathcal{U}_{\mathfrak{Q}}$ such that $A = \ulcorner a \urcorner$, that is $Y = \ulcorner a \urcorner$ if $\star = R$ and $Y = \{\ulcorner a \urcorner\}$ if $\star = F$.

Since $\eta^{\star}_{\mathfrak{Q}}$ preserves $\cdot$ and $\bigsqcup$, the closure of $\mathcal{P}^{\star}_{\mathbf{F}^{\star}(\mathbf{U}^{\star}(\mathfrak{Q}))} \cup \mathcal{U}^{\star}_{\mathbf{F}^{\star}(\mathbf{U}^{\star}(\mathfrak{Q}))}$ by compositions and unions is also included in the image of $\eta^{\star}_{\mathfrak{Q}}$. Then by minimality (15), the image of $\eta^{\star}_{\mathfrak{Q}}$ is equal to all of $\mathbf{F}^{\star}(\mathbf{U}^{\star}(\mathfrak{Q}))$, that is, $\eta^{\star}_{\mathfrak{Q}}$ is surjective.

Since $\eta^{\star}_{\mathfrak{Q}}$ is bijective and preserves the structure of ODAs, it is an UA-isomorphism. ■

5. Categorical Equivalence

In this section, we show that the work in the previous sections can be easily extended to a categorical equivalence. To be precise, we organize the class of complete orthomodular lattices and that of orthomodular dynamic algebras into two categories, respectively; the two constructions denoted by **F** and **U**, respectively, that we introduced before are shown to be functors between these two categories; finally, the bijections χ and η we used in the previous proofs are shown to be natural isomorphisms.

The categories we involve contain morphisms that are strong: the COL-morphisms are bijective (isomorphisms) and the ODA-morphisms are bijective on the induced projectors. While weaker morphisms may be suitable for each category (the category of orthomodular lattices tends to involve ortholattice homomorphisms that are not necessarily bijective), the presence of an inverse for each of our COL-morphisms allows us to conjugate by the morphism, which is widely used in our proofs. We leave for future work a generalization of categorical equivalence to weaker morphisms.

We formally define two categories of orthomodular dynamic algebras and the category of complete orthomodular lattices with isomorphisms.

Category of complete orthomodular lattices. Definitions of complete orthomodular lattices and ortholattice isomorphisms are reviewed in Section 2.1. It is a straightforward exercise to see that complete orthomodular lattices equipped with ortho-lattice isomorphisms form a category $\mathbb{L}$.

Categories of orthomodular dynamic algebras. We defined both F-ODAs and R-ODAs in 2.2. For both types of orthomodular dynamic algebra, we define their morphisms as follows.

DEFINITION 5.1. Let $\mathfrak{Q}_1 = (Q_1, \bigsqcup_1, \cdot_1, \sim_1)$ to $\mathfrak{Q}_2 = (Q_2, \bigsqcup_2, \cdot_2, \sim_2)$ be both R-ODAs (both F-ODAs). A function $\theta : Q_1 \to Q_2$ is a $\mathbb{Q}^F$-*morphism* ($\mathbb{Q}^R$-*morphism*) if:

35. θ restricted to $\mathcal{P}_1 = \{\sim x \mid x \in Q_1\}$ is an ortho-lattice isomorphism from $(\mathcal{P}_1, \preceq_1, \sim_1)$ to $(\mathcal{P}_2, \preceq_2, \sim_2)$;
36. θ preserves $\sim$, $\bigsqcup$ and $\cdot$, i.e. for any $A_1 \subseteq \mathcal{Q}_1$ and $x_1, y_1 \in \mathcal{Q}_1$, $\sim_2\theta(x_1) = \theta(\sim_1 x_1) \quad \theta(\bigsqcup_1 A_1) = \bigsqcup_2 \theta[A_1] \quad \theta(x_1 \cdot_1 y_1) = \theta(x_1) \cdot_2 \theta(x_2)$

THEOREM 5.2. *R-ODAs (F-ODAs) equipped with $\mathbb{Q}$-morphisms form a category $\mathbb{Q}^R$ ($\mathbb{Q}^F$).*

PROOF. Let $\mathbb{Q}$ be either $\mathbb{Q}^R$ or $\mathbb{Q}^F$. Let the identity arrows be the identity maps, and arrow composition be function compositions.

It is obvious that the identity maps are $\mathbb{Q}$-morphisms, and function compositions of $\mathbb{Q}$-morphisms are still $\mathbb{Q}$-morphisms and satisfy associativity.

■

For $\star \in \{R, F\}$, a $\mathbb{Q}^\star$-*isomorphism* is a bijective $\mathbb{Q}^\star$-morphism.

PROPOSITION 5.3. *For $\star \in \{R, F\}$ and $\star$-ODAs $\mathfrak{Q}_1 = (Q_1, \bigsqcup_1, \cdot_1, \sim_1)$ and $\mathfrak{Q}_2 = (Q_1, \bigsqcup_2, \cdot_2, \sim_2)$, a map from Q_1 to Q_2 is a $\mathbb{Q}^\star$-isomorphism if and only if it is a UA-isomorphism.*

PROOF. Let $f : Q_1 \to Q_2$. If f is a $\mathbb{Q}^\star$-isomorphism, then it is a bijection while also preserving the basic operators, and is hence a UA-isomorphism. Conversely, if f is a UA-isomorphism, we must show that acting on $\mathcal{P}$, it is an ortholattice isomorphism. We first observe that $f(p) \in \mathcal{P}_2$ when $p \in \mathcal{P}_1$, since if $p = \sim x$, then $f(p) = f(\sim x) = \sim f(x) \in \mathcal{P}_2$. Since f is bijective from Q_1 to Q_2 it is injective on any subset of Q_1, namely $\mathcal{P}_1$. To see that f is surjective, suppose that $q \in \mathcal{P}_2$. Then $q = \sim y$ for some $y \in Q_2$. By surjectivity of f onto Q_2, there is $x \in Q_1$ such that $f(x) = y$. Then

$f(\sim x) = \sim f(x) = \sim y = q$. Since f preserves $\sim$ and $\bigsqcup$, it also preserves $\bigvee$, and hence is an ortholattice isomorphism from $\mathcal{P}_1$ to $\mathcal{P}_2$. ■

Recall that for $\star \in \{R, F\}$, $\eta^\star_{\mathfrak{Q}}$ is a UA-isomorphism. Thus we have the following corollary:

COROLLARY 5.4. *For both $\star \in \{R, F\}$ and each $\star$-ODA $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$, $\eta^\star_{\mathfrak{Q}}$ is a $Q^\star$-isomorphism from $\mathfrak{Q}$ to $\mathbf{F}^\star(\mathbf{U}^\star(\mathfrak{Q}))$.*

5.1. From COLs to ODAs

We define functors $\mathbf{F}^F$ and $\mathbf{F}^R$ from the category of complete orthomodular lattices with isomorphisms to the category of F-ODAs and R-ODAs respectively. On objects, given a complete orthomodular lattice $\mathfrak{L}$, $\mathbf{F}^F(\mathfrak{L})$ and $\mathbf{F}^R(\mathfrak{L})$ are defined according to the construction given in Section 3.
Mapping of Arrows. Fix an $\mathbb{L}$-morphism k from $\mathfrak{L}_1 = (L_1, \leq_1, -^{\perp_1})$ to $\mathfrak{L}_2 = (L_2, \leq_2, -^{\perp_2})$. Define

$$\mathbf{F}^R(k) : \mathcal{Q}^R_1 \to \mathcal{Q}^R_2 :: A_1 \mapsto k \circ A_1 \circ k^{-1} \quad \text{(relation composition)}$$
$$\mathbf{F}^F(k) : \mathcal{Q}^F_1 \to \mathcal{Q}^F_2 :: A_1 \mapsto \{k \circ a_1 \circ k^{-1} \mid a_1 \in A_1\}$$

THEOREM 5.5. *For $\star \in \{R, F\}$, $\mathbf{F}^\star(k)$ is a $\mathbb{Q}$-morphisms from $\mathbf{F}^\star(\mathfrak{L}_1)$ to $\mathbf{F}^\star(\mathfrak{L}_2)$.*

PROOF. First, we observe that $\mathbf{F}^\star(k)$ is an ortho-lattice isomorphism from $(\mathcal{P}^\star_1, \preceq^\star_1, \overset{\star}{\sim}_1)$ to $(\mathcal{P}^\star_2, \preceq^\star_2, \overset{\star}{\sim}_2)$. By Proposition 3.4, $\mathcal{P}^R_1 = \{f_p \mid p \in L_1\}$, $\mathcal{P}^F_1 = \{\{f_p\} \mid p \in L_1\}$, $\mathcal{P}^R_2 = \{f_p \mid p \in L_2\}$ and $\mathcal{P}^F_2 = \{\{f_p\} \mid p \in L_2\}$, and χ^R_1, χ^F_1, χ^R_2, and χ^F_2 are defined. Note that $\mathbf{F}^\star(k) = \chi^\star_2 \circ k \circ (\chi^\star_1)^{-1}$ on $\mathcal{P}^\star_1$; because, for each $f_p \in \mathcal{P}^R_1$ and $\{f_p\} \in \mathcal{P}^F_1$, by Lemma 2.4 $f_{k(p)} \circ k = k \circ f_p$ and thus

$$(\chi^R_2 \circ k \circ (\chi^R_1)^{-1})(f_p) = \chi^R_2(k(p)) = f_{k(p)} = k \circ f_p \circ k^{-1} = \mathbf{F}^R(k)(f_p),$$
$$(\chi^F_2 \circ k \circ (\chi^F_1)^{-1})(\{f_p\}) = \chi^F_2(k(p)) = \{f_{k(p)}\} = \{k \circ f_p \circ k^{-1}\} = \mathbf{F}^F(k)(\{f_p\})$$

Since all of $\chi^\star_2$, k and $(\chi^\star_1)^{-1}$ are ortho-lattice isomorphisms, so is $\mathbf{F}^\star(k)$ on $\mathcal{P}^\star_1$.

Next we show that $\mathbf{F}^\star(k)$ preserves orthocomplementation. Let k be an ortholattice isomorphism from $\mathfrak{L}_1$ to $\mathfrak{L}_2$. Suppose $\mathfrak{L}_1$ has top element $\mathbb{1}_1$ and $\mathfrak{L}_2$ has top element $\mathbb{1}_2$. Then for any $x \in Q_1$,

$$\begin{aligned}\overset{\mathrm{R}}{\sim}\mathbf{F}^R(k)(x) &= \overset{\mathrm{R}}{\sim}(k \circ x \circ k^{-1}) = f_{(\bigvee\{y \mid (\mathbb{1}_2, y) \in k \circ x \circ k^{-1}\})^\perp} \\ &= f_{(k(\bigvee\{y \mid (\mathbb{1}_1, y) \in x\}))^\perp} = f_{k((\bigvee\{y \mid (\mathbb{1}_1, y) \in x\})^\perp)} \\ &= k \circ f_{(\bigvee\{y \mid (\mathbb{1}_1, y) \in x\})^\perp} \circ k^{-1} && \text{by Lemma 2.4}\end{aligned}$$

$$
\begin{aligned}
&= k \circ \overset{\mathrm{R}}{\sim} x \circ k^{-1} = \mathbf{F}^R(k)(\overset{\mathrm{R}}{\sim} x)\\
\overset{\mathrm{F}}{\sim} \mathbf{F}^R(k)(x) &= \overset{\mathrm{F}}{\sim}\{k \circ a \circ k^{-1} \mid a \in x\} = \{f_{(\bigvee\{(k\circ a\circ k^{-1})(\mathbb{1}_2) \mid a\in x\})^\perp}\}\\
&= \{f_{k((\bigvee\{a(\mathbb{1}_1)\mid a\in x\})^\perp)}\}\\
&= \left\{k \circ f_{(\bigvee\{a(\mathbb{1}_1)\mid a\in x\})^\perp} \circ k^{-1}\right\} && \text{by Lemma 2.4}\\
&= \mathbf{F}^F(k)\left(\{f_{(\bigvee\{a(\mathbb{1}_1)\mid a\in x\})^\perp}\}\right) = \mathbf{F}^F(k)(\overset{\mathrm{F}}{\sim} x)
\end{aligned}
$$

Second, we show that $\mathbf{F}^\star(k)$ preserves $\bigcup$. Let $\{A^i \in \mathcal{Q}_1 \mid i \in I\}$ be arbitrary.

$$
\begin{aligned}
\mathbf{F}^R(k)(\bigcup\{A^i \mid i \in I\}) &= k \circ (\bigcup\{A^i \mid i \in I\}) \circ k^{-1}\\
&= \bigcup\{k \circ A^i \circ k^{-1} \mid i \in I\}\\
&= \bigcup\{\mathbf{F}^R(k)(A^i) \mid i \in I\}\\
\mathbf{F}^F(k)(\bigcup\{A^i \mid i \in I\}) &= \mathbf{F}^F(k)(\{a \in \mathcal{F}_1 \mid a \in A^i, \text{ for some } i \in I\})\\
&= \{k \circ a \circ k^{-1} \in \mathcal{F}_1 \mid a \in A^i, \text{ for some } i \in I\}\\
&= \bigcup\left\{\{k \circ a \circ k^{-1} \in \mathcal{F}_1 \mid a \in A^i\} \mid i \in I\right\}\\
&= \bigcup\{\mathbf{F}^F(k)(A^i) \mid i \in I\}
\end{aligned}
$$

Third, we show that $\mathbf{F}^\star(k)$ preserves $\cdot$. Let $A, B \in \mathcal{Q}_1$ be arbitrary.

$$
\begin{aligned}
\mathbf{F}^R(k)(A \cdot B) &= k \circ A \circ B \circ k^{-1}\\
&= k \circ A \circ k^{-1} \circ k \circ B \circ k^{-1}\\
&= \mathbf{F}^R(k)(A) \cdot \mathbf{F}^R(k)(B)\\
\mathbf{F}^F(k)(A \cdot B) &= \mathbf{F}^F(k)(\{a \circ b \mid a \in A, b \in B\})\\
&= \{k \circ a \circ b \circ k^{-1} \mid a \in A, b \in B\}\\
&= \{k \circ a \circ k^{-1} \circ k \circ b \circ k^{-1} \mid a \in A, b \in B\}\\
&= \mathbf{F}^F(k)(A) \cdot \mathbf{F}^F(k)(B)
\end{aligned}
$$

■

THEOREM 5.6. *For* $\star \in \{R, F\}$, $\mathbf{F}^\star$ *is a functor from* $\mathbb{L}$ *to* $\mathbb{Q}$.

PROOF. By Theorems 3.18 and 5.5, $\mathbf{F}^\star$ maps objects in $\mathbb{L}$ to objects in $\mathbb{Q}$, and maps arrows $k : \mathfrak{L}_1 \to \mathfrak{L}_2$ in $\mathbb{L}$ to arrows $\mathbf{F}^\star(k) : \mathbf{F}^\star(\mathfrak{L}_1) \to \mathbf{F}^\star(\mathfrak{L}_2)$ in $\mathbb{Q}$.

It is easy to see from the definition that $\mathbf{F}^\star$ preserves the identity arrows. For arrow compositions, let $k : \mathfrak{L}_1 \to \mathfrak{L}_2$ and $l : \mathfrak{L}_2 \to \mathfrak{L}_3$ be two arbitrary $\mathbb{L}$-morphisms. Then $\mathbf{F}^\star(l) \circ \mathbf{F}^\star(k) = \mathbf{F}^\star(l \circ k)$ holds, because for any $A_1 \in \mathcal{Q}_1$

$$\begin{aligned}(\mathbf{F}^R(l) \circ \mathbf{F}^R(k))(A_1) &= \mathbf{F}^R(l)\big(k \circ A_1 \circ k^{-1}\big)\\ &= l \circ (k \circ A_1 \circ k^{-1}) \circ l^{-1}\\ &= (l \circ k) \circ A_1 \circ (l \circ k)^{-1}\\ &= \mathbf{F}^R(l \circ k)(A_1)\\ (\mathbf{F}^F(l) \circ \mathbf{F}^F(k))(A_1) &= \mathbf{F}^F(l)\big(\{k \circ a_1 \circ k^{-1} \mid a_1 \in A_1\}\big)\\ &= \{l \circ (k \circ a_1 \circ k^{-1}) \circ l^{-1} \mid a_1 \in A_1\}\\ &= \{(l \circ k) \circ a_1 \circ (l \circ k)^{-1} \mid a_1 \in A_1\}\\ &= \mathbf{F}^F(l \circ k)(A_1)\end{aligned}$$

■

5.2. From ODAs to COLs

In this subsection, we define two functors $\mathbf{U}^R : \mathbb{Q}^R \to \mathbb{L}$ and $\mathbf{U}^F : \mathbb{Q}^F \to \mathbb{L}$.
Mapping of Objects. For both $\star \in \{R, F\}$ and orthomodular dynamic algebra $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ in $\mathbb{Q}^\star$, define $\mathbf{U}^\star(\mathfrak{Q}) = \mathbf{U}(\mathfrak{Q}) = (\mathcal{P}, \preceq, \sim)$.

Theorem 5.7. *For $\star \in \{R, F\}$, $\mathbf{U}^\star(\mathfrak{Q})$ is a complete orthomodular lattice.*

Proof. It follows directly from the definition of orthomodular dynamic algebras. ■

Mapping of Arrows. Fix a $\mathbb{Q}^\star$-morphism $\theta : \mathfrak{Q}_1 \to \mathfrak{Q}_2$. Let $\mathbf{U}^\star(\theta)$ be the restriction of θ to $\mathcal{P}_1$.

Theorem 5.8. *For $\star \in \{R, F\}$, $\mathbf{U}^\star(\theta)$ is an $\mathbb{L}$-morphism from $\mathbf{U}^\star(\mathfrak{Q}_1)$ to $\mathbf{U}^\star(\mathfrak{Q}_2)$.*

Proof. This follows directly from the definition of $\mathbb{Q}^\star$-morphisms. ■

Theorem 5.9. *For $\star \in \{R, F\}$, $\mathbf{U}^\star$ is a functor from $\mathbb{Q}^\star$ to $\mathbb{L}$.*

Proof. By Theorems 5.7 and 5.8 $\mathbf{U}^\star$ maps objects in $\mathbb{Q}^\star$ to objects in $\mathbb{L}$, and maps arrows $\theta : \mathfrak{Q}_1 \to \mathfrak{Q}_2$ in $\mathbb{Q}^\star$ to arrows $\mathbf{U}^\star(\theta) : \mathbf{U}^\star(\mathfrak{Q}_1) \to \mathbf{U}^\star(\mathfrak{Q}_2)$ in $\mathbb{L}$. It is obvious from the definition that $\mathbf{U}^\star$ preserves the identity arrows and arrow composition. ■

5.3. The Natural Isomorphisms $\boldsymbol{\tau}^\star : \mathbf{1}_{\mathbb{L}} \longrightarrow \mathbf{U}^\star \circ \mathbf{F}^\star$

For each complete orthomodular lattice $\mathfrak{L} = (L, \leq, -^\perp)$ and $\star \in \{R, F\}$, we let $\tau^\star_{\mathfrak{L}} = \chi^\star$ which is defined in (31) and (32).

THEOREM 5.10. *For both* $\star \in \{R, F\}$, $\tau^\star : 1_{\mathbb{L}} \to \mathbf{U}^\star \circ \mathbf{F}^\star$ *is a natural isomorphism.*

PROOF. For each object $\mathfrak{L}$ in $\mathbb{L}$, by Proposition 3.4, $\tau^\star_{\mathfrak{L}} = \chi^\star$ is a bijective $\mathbb{L}$-morphism, so $\tau^\star_{\mathfrak{L}}$ has an inverse in $\mathbb{L}$ and is an isomorphism in $\mathbb{L}$.

For naturality, assume that $k : \mathfrak{L}_1 \to \mathfrak{L}_2$ is an $\mathbb{L}$-morphism. For each $p \in L_1$,

$$\begin{aligned}((\mathbf{U}^R \circ \mathbf{F}^R)(k) \circ \tau^R_{\mathfrak{L}_1})(p) &= (\mathbf{U}^R \circ \mathbf{F}^R)(k)(f_p) = \mathbf{U}^R(\mathbf{F}^R(k))(f_p)\\ &= \mathbf{F}^R(k)(f_p) = k \circ f_p \circ k^{-1}\\ &= f_{k(p)} && \text{by Lemma 2.4}\\ &= (\tau^R_{\mathfrak{L}_2} \circ k)(p)\\ ((\mathbf{U}^F \circ \mathbf{F}^F)(k) \circ \tau^F_{\mathfrak{L}_1})(p) &= (\mathbf{U}^F \circ \mathbf{F}^F)(k)(\{f_p\}) = \mathbf{U}^R(\mathbf{F}^R(k))(\{f_p\})\\ &= \mathbf{F}^R(k)(\{f_p\}) = \{k \circ f_p \circ k^{-1}\}\\ &= \{f_{k(p)}\} && \text{by Lemma 2.4}\\ &= (\tau^R_{\mathfrak{L}_2} \circ k)(p)\end{aligned}$$

Therefore, for both $\star \in \{R, F\}$, $(\mathbf{U}^\star \circ \mathbf{F}^\star)(k) \circ \tau_{\mathfrak{L}_1} = \tau^\star_{\mathfrak{L}_2} \circ k$. ■

5.4. The Natural Isomorphisms $\boldsymbol{\eta}^\star : \mathbf{1}^\star_{\mathbb{Q}} \longrightarrow \mathbf{F}^\star \circ \mathbf{U}^\star$

For each $\star \in \{R, F\}$ and $\star$-ODA $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ and, we let $\eta^\star_{\mathfrak{Q}}$ be defined in the same way as in Section 4.

THEOREM 5.11. *For* $\star \in \{R, F\}$, $\eta^\star : 1_{\mathbb{Q}} \to \mathbf{F}^\star \circ \mathbf{U}^\star$ *is a natural isomorphism.*

PROOF. For each $\star \in \{R, F\}$ and $\star$-ODA $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$, by Theorem 4.2 $\eta^\star_{\mathfrak{Q}}$ is an isomorphism in $\mathbb{Q}^\star$. It remains to verify naturality.

Let $\theta : \mathfrak{Q}_1 \to \mathfrak{Q}_2$ be a $\mathbb{Q}^\star$-morphism.

First note that for any $a \in Q_1$, we have that

$$\theta \circ \ulcorner a \urcorner \circ \theta^{-1} = \ulcorner \theta(a) \urcorner \tag{37}$$

since for any $b \in \mathfrak{Q}_2$, we have that $\theta(\ulcorner a \urcorner(\theta^{-1}(b))) = \theta({\sim\sim}(a \cdot \theta^{-1}(b))) = {\sim\sim}\theta(a \cdot \theta^{-1}(b)) = {\sim\sim}(\theta(a) \cdot b) = \ulcorner \theta(a) \urcorner(b)$.

For each $x \in Q_1$,

$$\big((\mathbf{F}^R \circ \mathbf{U}^R)(\theta) \circ \eta^R_{\mathfrak{Q}_1}\big)(x) = \big((\mathbf{F}^R \circ \mathbf{U}^R)(\theta) \circ \eta^R_{\mathfrak{Q}_1}\big)\big(\bigsqcup\{y \mid y \in \mathcal{T}^R_1(x)\}\big)$$

$$
\begin{aligned}
&= (\mathbf{F}^R \circ \mathbf{U}^R)(\theta)(\bigcup\{\ulcorner y \urcorner \mid y \in \mathcal{T}_1^R(x)\}) \\
&= \bigcup\left\{(\mathbf{F}^R \circ \mathbf{U}^R)(\theta)(\ulcorner y \urcorner) \mid y \in \mathcal{T}_1^R(x)\right\} \\
&= \bigcup\left\{\ulcorner \theta(y) \urcorner \mid y \in \mathcal{T}_1^R(x)\right\} && \text{by (37)} \\
&= \eta^R_{\mathfrak{Q}_2}\left(\bigsqcup\{\theta(y) \mid y \in \mathcal{T}_1^R(x)\}\right) \\
&= (\eta^R_{\mathfrak{Q}_2} \circ \theta)\left(\bigsqcup\{y \mid y \in \mathcal{T}_1^R(x)\}\right) \\
&= (\eta^R_{\mathfrak{Q}_2} \circ \theta)(x)
\end{aligned}
$$

$$
\begin{aligned}
\big((\mathbf{F}^F \circ \mathbf{U}^F)(\theta) \circ \eta^F_{\mathfrak{Q}_1}\big)(x) &= \big((\mathbf{F}^F \circ \mathbf{U}^F)(\theta) \circ \eta^R_{\mathfrak{Q}_1}\big)\big(\bigsqcup\{y \mid y \in \mathcal{T}_1^F(x)\}\big) \\
&= (\mathbf{F}^F \circ \mathbf{U}^F)(\theta)\left(\{\ulcorner y \urcorner \mid y \in \mathcal{T}_1^F(x)\}\right) \\
&= \left\{\theta \circ \ulcorner y \urcorner \circ \theta^{-1} \mid y \in \mathcal{T}_1^F(x)\right\} \\
&= \left\{\ulcorner \theta(y) \urcorner \mid y \in \mathcal{T}_1^F(x)\right\} && \text{by (37)} \\
&= \eta^F_{\mathfrak{Q}_2}\left(\bigsqcup\{\theta(y) \mid y \in \mathcal{T}_1^F(x)\}\right) \\
&= (\eta^F_{\mathfrak{Q}_2} \circ \theta)\left(\bigsqcup\{y \mid y \in \mathcal{T}_1^F(x)\}\right) \\
&= (\eta^R_{\mathfrak{Q}_2} \circ \theta)(x)
\end{aligned}
$$

Therefore, $(\mathbf{F}^\star \circ \mathbf{U}^\star)(\theta) \circ \eta^\star_{\mathfrak{Q}_1} = \eta^\star_{\mathfrak{Q}_2} \circ \theta$. ∎

Finally, we arrive at the following conclusion.

THEOREM 5.12. *For* $\star \in \{R, F\}$, $(\mathbf{F}^\star, \mathbf{U}^\star, \tau^\star, \eta^\star)$ *forms a categorical equivalence between* $\mathbb{L}^\star$ *and* $\mathbb{Q}^\star$*.*

REMARK 5.13. Had we replaced *complete orthomodular lattice* in Definitions 2.9 and 13 with a more restricted type of lattice such as a Hilbert lattice or Piron lattice, we could then show these orthomodular dynamic algebras are categorically equivalent to the corresponding category of lattices (Hillbert lattice or Piron lattice with ortho-lattice isomorphisms) using the same functors and natural isomorphisms as we use here. The proof remains exactly the same. This thus establishes a link between orthomodular dynamic algebras and Hilbert spaces (or other important quantum structures) via these stronger lattice structures, particularly the Hilbert lattices.

REMARK 5.14. We now briefly look at how the R-ODAs and F-ODAs relate to each other. The functors $\mathbf{F}^*$ and $\mathbf{U}^*$ that we introduced above seem like natural candidates for establishing this connection. Using these we can define

$$\mathbf{G} := \mathbf{F}^R \circ \mathbf{U}^F : \mathbb{Q}^F \to \mathbb{Q}^R.$$

The functor $\mathbf{G}$ describes a natural and interesting connection for F-ODAs and R-ODAs constructed from a complete orthomodular lattice $\mathfrak{L}$. The elements of the F-ODA constructed from $\mathfrak{L}$, $\mathbf{F}^F(\mathfrak{L})$, are sets of functions constructed from $\mathfrak{L}$-automorphisms and $\mathfrak{L}$ projections. These naturally relate to the R-ODA, $\mathbf{F}^R(\mathfrak{L})$ whose elements are relations on $\mathfrak{L}$ by mapping a set of such functions to their set-theoretic union to get a relation. The functor $\mathbf{G}$ establishes precisely this connection:

Let $\mathfrak{L}$ be an orthomodular lattice and $\mathbf{F}^F(\mathfrak{L}) = (Q^F, \bigcup, \overset{F}{\cdot}, \overset{\mathrm{F}}{\sim})$. Then $\mathbf{G}(\mathbf{F}^F(\mathfrak{L})) = \mathbf{F}^R(\mathfrak{L}) = (Q^R, \bigcup, \overset{R}{\cdot}, \overset{\mathrm{R}}{\sim}) \in \mathbb{Q}^R$. In particular $Q^R = \{\bigcup A \mid A \in Q^F\}$.

To see this let $\mathfrak{L}$ be an orthomodular lattice and $\mathcal{F}$ be the smallest set containing Sasaki projections and $\mathfrak{L}$-automorphisms that is closed under function composition, as before. Then by construction we have $Q^F = \wp(\mathcal{F})$, and we have $\mathbf{U}^F(\mathbf{F}^F(\mathfrak{L})) = \mathfrak{L}_F = (\mathcal{P}_F, \preceq_F, \sim_F)$. By representation theorem we have $\mathfrak{L}_F = \mathfrak{L}$ and thus $\mathbf{G}(\mathbf{F}^F(\mathfrak{L})) = \mathbf{F}^R(\mathbf{U}^F(\mathbf{F}^F(\mathfrak{L})) = \mathbf{F}^R(\mathfrak{L}) = \mathbf{F}^R(\mathfrak{L})$.

6. Involutions on Orthomodular Dynamic Algebras

In this section, we show that orthomodular dynamic algebras can be naturally equipped with involutions, which generalize adjoints of projectors and unitaries on Hilbert spaces. Here we only investigate the defining properties of involutions and how they behave in the categorical equivalence. The subtle interplay between involution and the defining conditions of orthomodular dynamic algebras will be left for future work.

We start from the definition of an involution.

Definition 6.1. An *involution* on an R-ODA or an F-ODA $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ is a function $^\dagger : Q \to Q$ satisfying all of the following:

1. For each $x \in Q$, $x^{\dagger\dagger} = x$.
2. For each $x \in Q$, $(\sim x)^\dagger = \sim x$.
3. For any $x, y \in Q$, $(x \cdot y)^\dagger = y^\dagger \cdot x^\dagger$.
4. For each $X \subseteq Q$, $(\bigsqcup X)^\dagger = \bigsqcup\{x^\dagger \mid x \in X\}$.
5. For each $a \in \mathcal{U}$, $a^\dagger \cdot a = a \cdot a^\dagger = \mathbb{1}$.

Next we define an involution in a concrete orthomodular dynamic algebra constructed from a complete orthomodular lattice.

DEFINITION 6.2. Let $\mathfrak{L} = (L, \leq, -^{\perp})$ be a complete orthomodular lattice.
Define a function $^{\triangle} : \mathcal{F} \to \mathcal{F}$ recursively as follows:

1. For each ortholattice automorphism $a \in \mathcal{A}$, $a^{\triangle} \stackrel{\text{def}}{=} a^{-1}$.
2. For each $p \in L$, $f_p^{\triangle} \stackrel{\text{def}}{=} f_p$.
3. For any $x, y \in \mathcal{F}$, $(x \circ y)^{\triangle} \stackrel{\text{def}}{=} y^{\triangle} \circ x^{\triangle}$.

Define a function $^{\dagger} : \mathcal{Q}^R \to \mathcal{Q}^R$ such that: for each $A \subseteq \mathcal{F}$, $(\bigcup A)^{\dagger} \stackrel{\text{def}}{=} \bigcup\{x^{\triangle} \mid x \in A\}$.

Also define a function $^{\dagger} : \mathcal{Q}^F \to \mathcal{Q}^F$ such that: for each $A \subseteq \mathcal{F}$, $A^{\dagger} \stackrel{\text{def}}{=} \{x^{\triangle} \mid x \in A\}$.

A simple and useful result about the function $^{\triangle}$ is as follows:

LEMMA 6.3. *For each $x \in \mathcal{F}$, $x^{\triangle\triangle} = x$.*

PROOF. First note that, for each ortholattice automorphism $a \in \mathcal{A}$, $a^{-1} \in \mathcal{A}$, so $a^{\triangle\triangle} = (a^{-1})^{\triangle} = (a^{-1})^{-1} = a$. Second note that, for each $p \in L$, $f_p^{\triangle\triangle} = f_p^{\triangle} = f_p$. Third note that, for any $x, y \in \mathcal{F}$, if $x^{\triangle\triangle} = x$ and $y^{\triangle\triangle} = y$, then $(x \circ y)^{\triangle\triangle} = (y^{\triangle} \circ x^{\triangle})^{\triangle} = x^{\triangle\triangle} \circ y^{\triangle\triangle} = x \circ y$. The required result thus follows. ■

Now we are ready to show that the function † is indeed an involution. First we consider the case for R-ODAs.

PROPOSITION 6.4. *† is an involution on $\mathbf{F}^R(\mathfrak{L})$.*

PROOF. **For Item 1**: Take an arbitrary element in $\mathcal{Q}^R$; then it is of the form $\bigcup A$, for some $A \subseteq \mathcal{F}$. By definition and Lemma 6.3 $(\bigcup A)^{\dagger\dagger} = (\bigcup\{x^{\triangle} \mid x \in A\})^{\dagger} = \bigcup\{x^{\triangle\triangle} \mid x \in A\} = \bigcup\{x \mid x \in A\} = \bigcup A$.

For Item 2: Take an arbitrary element in $\mathcal{Q}^R$; then it is of the form $\bigcup A$, for some $A \subseteq \mathcal{F}$.

$$\begin{aligned}
\left({\sim}\bigcup A\right)^{\dagger} &= \left(f_{(\bigvee\{b|(\mathbb{1},b)\in A\})^{\perp}}\right)^{\dagger} \\
&= \left(\bigcup\{f_{(\bigvee\{b|(\mathbb{1},b)\in A\})^{\perp}}\}\right)^{\dagger} \\
&= \bigcup\left\{f^{\triangle}_{(\bigvee\{b|(\mathbb{1},b)\in A\})^{\perp}}\right\} \\
&= \bigcup\left\{f_{(\bigvee\{b|(\mathbb{1},b)\in A\})^{\perp}}\right\} \\
&= f_{(\bigvee\{b|(\mathbb{1},b)\in A\})^{\perp}} \\
&= {\sim}\bigcup A
\end{aligned}$$

For Item 3: Take two arbitrary elements in $\mathcal{Q}^R$; then they are of the form $\bigcup A$ and $\bigcup B$, for some $A, B \in \wp(\mathcal{F})$, respectively.

$$\begin{aligned}
\left(\bigcup A \cdot \bigcup B\right)^{\dagger} &= \left(\bigcup A \circ \bigcup B\right)^{\dagger} \\
&= \left(\bigcup \{a \circ b \mid a \in A, b \in B\}\right)^{\dagger} \\
&= \bigcup \{(a \circ b)^{\triangle} \mid a \in A, b \in B\} \\
&= \bigcup \{b^{\triangle} \circ a^{\triangle} \mid a \in A, b \in B\} \\
&= \bigcup \{b^{\triangle} \mid b \in B\} \circ \bigcup \{a^{\triangle} \mid a \in A\} \\
&= (\bigcup B)^{\dagger} \circ (\bigcup A)^{\dagger} \\
&= (\bigcup B)^{\dagger} \cdot (\bigcup A)^{\dagger}
\end{aligned}$$

For Item 4: Take an arbitrary set of elements in $\mathcal{Q}^R$; then it is of the form $\{\bigcup X \mid X \in \mathcal{X}\}$ for some $\mathcal{X} \subseteq \wp(\mathcal{F})$.

$$\begin{aligned}
\left(\bigsqcup \left\{\bigcup X \mid X \in \mathcal{X}\right\}\right)^{\dagger} &= \left(\bigcup \left\{\bigcup X \mid X \in \mathcal{X}\right\}\right)^{\dagger} \\
&= \left(\bigcup \left\{\bigcup \{a \mid a \in X\} \mid X \in \mathcal{X}\right\}\right)^{\dagger} \\
&= \left(\bigcup \{a \mid \text{there is an } X \in \mathcal{X} \text{ such that } a \in X\}\right)^{\dagger} \\
&= \bigcup \{a^{\triangle} \mid \text{there is an } X \in \mathcal{X} \text{ such that } a \in X\} \\
&= \bigcup \left\{\bigcup \{a^{\triangle} \mid a \in X\} \mid X \in \mathcal{X}\right\} \\
&= \bigcup \left\{\left(\bigcup X\right)^{\dagger} \mid X \in \mathcal{X}\right\} \\
&= \bigsqcup \left\{\left(\bigcup X\right)^{\dagger} \mid X \in \mathcal{X}\right\}
\end{aligned}$$

For Item 5: Let $a \in \mathcal{U}^R = \mathcal{A}$.

$$a^{\dagger} \cdot a = \left(\bigcup \{a\}^{\dagger} \circ \bigcup \{a\}\right) = \left(\bigcup \{a^{\triangle}\} \circ \bigcup \{a\}\right) = a^{\triangle} \circ a = a^{-1} \circ a = f_{\mathbb{1}}$$

$$a \cdot a^{\dagger} = \left(\bigcup \{a\} \circ \bigcup \{a\}^{\dagger}\right) = \left(\bigcup \{a\} \circ \bigcup \{a^{\triangle}\}\right) = a \circ a^{\triangle} = a \circ a^{-1} = f_{\mathbb{1}}$$

■

Second we consider the case for F-ODAs.

Proposition 6.5. † *is an involution on* $\mathbf{F}^F(\mathfrak{L})$.

PROOF. **For Item 1**: Take an arbitrary element in $\mathcal{Q}^F$; then it is of the form A, for some $A \subseteq \mathcal{F}$. By definition and Lemma 6.3 $A^{\dagger\dagger} = \{x^{\triangle} \mid x \in A\}^{\dagger} = \{x^{\triangle\triangle} \mid x \in A\} = \{x \mid x \in A\} = A$.

For Item 2: Take an arbitrary element in $\mathcal{Q}^F$; then it is of the form A, for some $A \subseteq \mathcal{F}$.

$$\begin{aligned}
(\sim A)^{\dagger} &= \left\{f_{(\bigvee\{b \mid (\mathbb{1},b)\in A\})^{\perp}}\right\}^{\dagger} \\
&= \left\{f^{\triangle}_{(\bigvee\{b \mid (\mathbb{1},b)\in A\})^{\perp}}\right\} \\
&= \left\{f_{(\bigvee\{b \mid (\mathbb{1},b)\in A\})^{\perp}}\right\} \\
&= \sim A
\end{aligned}$$

For Item 3: Take two arbitrary elements in $\mathcal{Q}^F$; then they are of the form A and B, for some $A, B \in \wp(\mathcal{F})$, respectively.

$$\begin{aligned}
(A \cdot B)^{\dagger} &= \big(\{a \mid a \in A\} \cdot \{b \mid b \in B\}\big)^{\dagger} \\
&= (\{a \circ b \mid a \in A, b \in B\})^{\dagger} \\
&= \{(a \circ b)^{\triangle} \mid a \in A, b \in B\} \\
&= \{b^{\triangle} \circ a^{\triangle} \mid a \in A, b \in B\} \\
&= \{b^{\triangle} \mid b \in B\} \cdot \{a^{\triangle} \mid a \in A\} \\
&= B^{\dagger} \cdot A^{\dagger}
\end{aligned}$$

For Item 4: Take an arbitrary set of elements in $\mathcal{Q}^F$; then it is of the form $\{X \mid X \in \mathcal{X}\}$ for some $\mathcal{X} \subseteq \wp(\mathcal{F})$.

$$\begin{aligned}
\Big(\bigsqcup\{X \mid X \in \mathcal{X}\}\Big)^{\dagger} &= \Big(\bigcup\{X \mid X \in \mathcal{X}\}\Big)^{\dagger} \\
&= \Big(\bigcup\big\{\{a \mid a \in X\} \mid X \in \mathcal{X}\big\}\Big)^{\dagger} \\
&= \Big(\bigcup\{a \mid \text{there is an } X \in \mathcal{X} \text{ such that } a \in X\}\Big)^{\dagger} \\
&= \bigcup\big\{a^{\triangle} \mid \text{there is an } X \in \mathcal{X} \text{ such that } a \in X\big\} \\
&= \bigcup\big\{\{a^{\triangle} \mid a \in X\} \mid X \in \mathcal{X}\big\} \\
&= \bigcup\big\{X^{\dagger} \mid X \in \mathcal{X}\big\} \\
&= \bigsqcup\big\{X^{\dagger} \mid X \in \mathcal{X}\big\}
\end{aligned}$$

For Item 5: Take an arbitrary element in $\mathcal{U}^F$; then it is of the form $\{a\}$, for some $a \in \mathcal{A}$.

$$\{a\}^\dagger \cdot \{a\} = \big(\{a\}^\dagger \circ \{a\}\big) = \{a^\triangle\} \circ \{a\} = \{a^{-1}\} \circ \{a\} = \{a^{-1} \circ a\} = \{f_{\mathbb{1}}\}$$
$$\{a\} \cdot \{a\}^\dagger = \big(\{a\} \circ \{a\}^\dagger\big) = \{a\} \circ \{a^\triangle\} = \{a\} \circ \{a^{-1}\} = \{a \circ a^{-1}\} = \{f_{\mathbb{1}}\}$$

■

Finally, we show that involutions are preserved by the natural isomorphism η.

We start from a lemma.

LEMMA 6.6. *Let $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ be an R-ODA or an F-ODA and † an involution on $\mathfrak{Q}$. For each $x \in \mathcal{T}$, $\ulcorner x^\dagger \urcorner = \ulcorner x \urcorner^\triangle$.*

PROOF. First note that, for any $a \in \mathcal{U}$ and $p \in \mathcal{P}$, by (19)

$$\begin{aligned}\ulcorner a \urcorner(\ulcorner a^\dagger \urcorner(p)) &= \ulcorner a \urcorner(\sim\sim(a^\dagger \cdot p)) = \ulcorner a \urcorner(a^\dagger \cdot p) = \sim\sim(a \cdot a^\dagger \cdot p)\\ &= \sim\sim(\mathbb{1} \cdot p) = \sim\sim p = p\\ \ulcorner a^\dagger \urcorner(\ulcorner a \urcorner(p)) &= \ulcorner a^\dagger \urcorner(\sim\sim(a \cdot p)) = \ulcorner a^\dagger \urcorner(a \cdot p) = \sim\sim(a^\dagger \cdot a \cdot p)\\ &= \sim\sim(\mathbb{1} \cdot p) = \sim\sim p = p\end{aligned}$$

so $\ulcorner a^\dagger \urcorner = \ulcorner a \urcorner^{-1} = \ulcorner a \urcorner^\triangle$. Second note that, for each $p \in \mathcal{P}$, $\ulcorner p^\dagger \urcorner = \ulcorner p \urcorner = \ulcorner p \urcorner^\triangle$. Third note that, for any $x, y \in \mathcal{T}$, if $\ulcorner x^\dagger \urcorner = \ulcorner x \urcorner^\triangle$ and $\ulcorner y^\dagger \urcorner = \ulcorner y \urcorner^\triangle$, then

$$\ulcorner (x \cdot y)^\dagger \urcorner = \ulcorner y^\dagger \cdot x^\dagger \urcorner = \ulcorner y^\dagger \urcorner \circ \ulcorner x^\dagger \urcorner = \ulcorner y \urcorner^\triangle \circ \ulcorner x \urcorner^\triangle = (\ulcorner x \urcorner \circ \ulcorner y \urcorner)^\triangle = \ulcorner x \cdot y \urcorner^\triangle$$

The required result thus follows. ■

First we consider the case for R-ODAs.

PROPOSITION 6.7. *Let $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ be an R-ODA and † an involution on $\mathfrak{Q}$. For each $x \in Q$, $\eta^R_{\mathfrak{Q}}(x^\dagger) = \eta^R_{\mathfrak{Q}}(x)^\dagger$.*

PROOF. Let $x \in Q$ be arbitrary. By the normal form theorem $x = \bigsqcup\{x_i \mid i \in I\}$, for some $\{x_i \mid i \in I\} \subseteq \mathcal{T}$. Then

$$\begin{aligned}\eta^R_{\mathfrak{Q}}(x^\dagger) &= \eta^R_{\mathfrak{Q}}((\bigsqcup\{x_i \mid i \in I\})^\dagger)\\ &= \eta^R_{\mathfrak{Q}}(\bigsqcup\{x_i^\dagger \mid i \in I\})\\ &= \bigcup\{\ulcorner x_i^\dagger \urcorner \mid i \in I\}\\ &= \bigcup\{\ulcorner x_i \urcorner^\triangle \mid i \in I\} \quad \text{(by Lemma 6.6)}\\ &= (\bigcup\{\ulcorner x_i \urcorner \mid i \in I\})^\dagger\end{aligned}$$

$$= \eta^R_{\mathfrak{Q}}(x)^\dagger$$

■

Second we consider the case for F-ODAs.

Proposition 6.8. *Let* $\mathfrak{Q} = (Q, \bigsqcup, \cdot, \sim)$ *be an F-ODA and* † *an involution on* $\mathfrak{Q}$. *For each* $x \in Q$, $\eta^F_{\mathfrak{Q}}(x^\dagger) = \eta^F_{\mathfrak{Q}}(x)^\dagger$.

Proof. Let $x \in Q$ be arbitrary. By the normal form theorem $x = \bigsqcup\{x_i \mid i \in I\}$, for some $\{x_i \mid i \in I\} \subseteq \mathcal{T}$. Then

$$\begin{aligned}
\eta^F_{\mathfrak{Q}}(x^\dagger) &= \eta^F_{\mathfrak{Q}}((\bigsqcup\{x_i \mid i \in I\})^\dagger) \\
&= \eta^F_{\mathfrak{Q}}(\bigsqcup\{x_i^\dagger \mid i \in I\}) \\
&= \{\ulcorner x_i^\dagger \urcorner \mid i \in I\} \\
&= \{\ulcorner x_i \urcorner^\triangle \mid i \in I\} \quad \text{(by Lemma 6.6)} \\
&= (\{\ulcorner x_i \urcorner \mid i \in I\})^\dagger \\
&= \eta^F_{\mathfrak{Q}}(x)^\dagger
\end{aligned}$$

■

7. Conclusion

This paper expands the role of orthomodular dynamic algebras to handle unitary operators. It also shows how an ODA can be constructed from a COL using relations rather than sets of functions and clarifies how the resulting ODA is distinct from the one constructed as sets of functions. We provide a representation theorem to show how, in a non-categorical setting, both types of ODAs are effectively the same as COLs. To strengthen this link, we put both ODAs and COLs in a categorical setting, adding morphisms and establishing a categorical equivalence among these categories.

The morphisms we use are bijective, which are more restrictive than the typical morphism used for ortholattices. The use of bijective morphisms seems essential for our proof strategies (we rely on conjugation by morphisms), but we imagine there may be a different approach that could allow the morphisms to be weakened. But we leave this for future exploration.

8. Coda: Quantale-modules

Quantales can be put into action by acting on a module as in [1]. We may view the elements of a quantale as a structure of operations acting on static properties of a computation organized as a module. A module over a quantale (aka quantale-module), discussed in [1], is a sup-lattices together with an operation that takes an element of the module together with an element of the quantale and outputs another element of the module. A basic example of a quantale-module is a quantale acting on itself (see [1]). We can define ODA-modules by maintaining essentially the same definition as a quantale-module, and we observe that a $\star$-ODA is a module over itself. A simpler example is the collection of subsets of a COL over the $\star$-ODA the COL generates. Using these examples, we may more specifically think of the atoms of a $\star$-ODA as deterministic actions, and the atoms of the module may be deterministic properties of the computation. Likewise, a non-atom in the $\star$-ODA would be a non-deterministic action and may transform an atom in the module to an element of the module that is not an atom, thus creating a non-deterministic property. Future work may develop this perspective further and develop examples in quantum computation. Future work may also lead to a logic on such ODA-modules that describes their quantum dynamics.

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